Mathematics Framework
Appendix A: Key Mathematical Ideas to Promote Student Success in Introductory University Courses in Quantitative Fields

Key Mathematical Ideas to Promote Student Success in Introductory University Courses in Quantitative Fields[1]

One of the important goals of K–12 mathematics is to prepare students for success in quantitative majors in college, should they choose to follow such a path. The route to equity in college-level education lies in good high school preparation. For a high school math pathway to provide good preparation for a major, it needs to include the cumulative math knowledge and mathematical ways of thinking that are assumed in introductory courses for that major. Many foundational courses in quantitative majors require either calculus or topics and precise, rigorous ways of thinking that are currently often learned on the path to calculus (e.g., facility with functions and algebra that arise in statistics). Moreover, students whose majors require calculus need to be prepared to learn it in college if they have not done so in high school. Educational developments of recent decades offer new ways to effectively deliver math curricula. These include group work, active learning, and certain kinds of classroom technology. Motivation inspired by applications has always been a component of good mathematics teaching. Today, engaging contexts for many high school math topics can be drawn from business, computer science, data science, social sciences and even computer gaming design, complementing traditional applications from the natural sciences and finance.

To promote success in introductory university courses in quantitative fields, students should be exposed to key mathematical ideas. The list below focuses on topics prior to calculus; the order of the items has no significance. The final item (complex numbers) has an asterisk because it is of tremendous importance in some quantitative fields

---

[1] From comments submitted by Patrick Callahan (Callahan Consulting), Brian Conrad (Professor, Department of Mathematics, Stanford University), and Rafe Mazzeo (Professor, Department of Mathematics, Stanford University).

(chemistry, engineering, physics, math) but not others (e.g., biology and economics). Students should:

1. **Understand representations of functions.** Functions as input-output laws can be expressed in many ways: algebraically as a formula or as a recursive formula (such as for the factorial function); visually as a graph or a table of values; and so on. Exposure to the many ways of describing a function makes the concept more tangible (e.g., relating the graphs of $f(3x)$, $f(x+4)$, and $-2f(x)$ to that of $f(x)$) and helps students to grasp the broad importance and ubiquity of functions throughout mathematics and their applications. Computer programming uses functions extensively, as do science, finance, engineering, and statistics.

2. **Be familiar with a variety of functions and manipulations with them.** Included here are linear functions, the absolute value, polynomials, rational functions (relating back to comfort with fractions), exponential functions a^x, logarithmic functions $\log_b(x)$, and trigonometric functions (especially $\sin x$, $\cos x$, $\tan x$). Students should know the basic shape of the functions' graphs (e.g., a line for $2x-7$, periodic vertical asymptotes for $\tan x$, and how the graphs of x^2 and x^3 and 2^x differ) and have a sense of their orders of magnitude (linear versus x^7 versus 2^x or $\log(x)$) as well as of special algebraic rules for exponentials and logarithms. These topics provide further opportunities to reinforce algebra material.

3. **Be familiar with modeling with functions.** A particular mathematical model can be used and re-used to answer many quantitative questions. This reusability property is why mathematical models are worth formulating. Translating words into equivalent mathematical expressions and equations, along with the reverse process, are fundamental to all uses of math to solve problems in the real world. This translation capability provides validation of a student's grasp of the meaning of mathematical concepts. It also represents a different way of thinking from that required by other, more self-contained mathematical concepts. Translation capability needs to be developed over time. It is not fully mastered before arrival in college but should be practiced at progressive levels of complexity, starting very early in a student's education.

When expressing the information from a word problem in terms of mathematics, an often essential step is to introduce an appropriate function and to clarify hypotheses and definitions. Examples include exponentials for understanding pandemics and investment (geometric growth), logarithms for visualizing data that span many orders of magnitude, and sines and cosines to model periodic phenomena (giving contexts far beyond triangles for the relevance of such functions; the addition laws for sine and cosine encode phase-shifting). Data science, natural sciences, and computer science provide numerous examples of modeling with many types of functions. Even if a student won’t use a specific class of functions later on, exposure in high school to a wealth of function types and their utility gives the overall concept a firm grounding in reality. Students should also see how a mathematical model can be re-used to solve problems beyond the initial one that gave rise to the model (e.g., bankers and customers use the same compound interest model to answer different questions).

4. Know the importance of focusing carefully enough on details to demonstrate good meta-cognition and to arrive at a reliable answer. Being self-critical and always checking for consistency are important habits for the reliable application of mathematics and are only acquired through experience in solving mathematical problems. These habits include finding one's mistake(s) when something has gone awry and checking that an answer “makes sense" in basic ways (e.g., an area cannot be negative; if a bank account is earning interest, then its value later should be larger). The use of technology does not eliminate the need for the latter, since erroneous information can be put into a computer. It is important to develop a sense of when an answer delivered by technological means is way off base, signifying that the input was incorrect.

Mathematical problems can often be solved in a variety of ways, and it is both legitimate and important to often allow students the choice of solving problems in ways that make the most sense to them. However, an essential feature of mathematics is the concept of “correct answer" (in the sense of the outcome of a calculation or solving equation(s) reliably) alongside attention to solution methods. The learning of mathematics should not emphasize “answer finding” to the exclusion of understanding of methodology, but the idea that many mathematical problems have a unique answer is important in many applications of mathematical models. Students should know that different exact solution methods *always* arrive at the same answer when no mistakes have been made and hypotheses remain unchanged and that different approximate methods arrive at nearby answers. (Two collections of data in a mathematical model often differ, but measuring data is not solving an exact mathematical problem.)

The internal consistency of math is stronger than what is encountered in other areas of life, and that consistency is crucial for the reliability of engineering, the development and analysis of mathematical models, and the writing and trouble-shooting of computer programs. Scientific advances and modern technologies now taken for granted (e.g., accurate GPS in planes and cars) rely crucially on mathematical problems having a "correct answer," and students should appreciate this consistency inherent in mathematics.

5. Be familiar and comfortable with symbolic manipulation, skills reinforced and extended in coursework after a first algebra course. A reasonable level of comfort and confidence in the reading and manipulation of symbolic expressions is absolutely essential for reliable work with mathematics (even when using a computer to do number-crunching). This does not mean grappling with very complicated expressions, but rather is about reaching a level of comfort with symbolic expressions, applying basic manipulations with confidence, and knowing what one is doing with algebra and why one is doing it.

Examples include being able to work correctly with fractions (e.g., dividing one fraction by another and reassembling as a single fraction), to read an algebraic expression (correctly interpreting the order of operations), to plug in numbers for symbols to get numerical output, and to manipulate symbolic expressions in accordance with the laws of algebra—that is, to be comfortable with simplifying, factoring, working with square roots and exponents (e.g., express a ratio of powers of a common number as a single power of that number), etc. Students should also understand how to manipulate inequalities (such as in problems involving constraints or optimization).

Certain facts with whole numbers, such as $a(b+c) = ab + ac$ and $a^{n+m} = a^n a^m$, remain valid for broader types of numbers (negative, rational, and real). This wider validity should be highlighted, so that students are aware of and become comfortable with its reliability. It is less important to know the names of such rules than it is to be aware of what rules are true; this is the mathematical counterpart of learning how to spell words. The laws of algebra should be seen as summarizing and abstracting facts from extensive concrete numerical experience, and not as arbitrary rules out of thin air to be memorized by rote. Indeed, students should learn that there is an inevitability to these rules, and that memorization is usually the least effective way to work with them. Developing this facility goes hand-in-hand with recognizing the falsity, in general, of statements such as “$(a+b)/(a+c) = b/c$" or passing sums through square roots or powers or absolute values. Plugging small numbers into a potential symbolic equality should be instinctive as a safety check (not as a justification, but as a way of sniffing out generally false statements).

6. Be able to work with and solve equations (and inequalities). This includes solving linear and quadratic equations in one variable, solving two linear equations in two unknowns, adding and subtracting equations from each other, and understanding the visual meaning of such problems (crossing of two lines at a point, or finding where a graph crosses the horizontal coordinate axis). Solving exponential equations using logarithms is another important class of examples, as is knowing that often inequalities can be solved using analogues of methods for solving equations (along with some case-by-case work).

The key principle is to avoid a zoological chart of types of equations, and provide students with the means to gain experience using manipulation of both sides of an equation to isolate an unknown quantity to solve for it, and then (when relevant) to interpret the answer. It is important to be aware that sometimes an equation has no solutions or multiple solutions, and to know what that means in terms of a mathematical model. It is likewise important to recognize that an equation may be expressed in many equivalent forms (by applying a reversible operation to both sides).

7. Understand the mathematics of measurement. This includes algebraic work with units of measurement (e.g., conversion among different units, using kilometers per hour, and recognizing that it makes no sense to add quantities with different units of measurement) and the development of an instinct to always use dimensional analysis (e.g., one cannot add a quantity measured in inches to one measured in square inches). Other crucial relevant skills include using ratios and percentages (as useful alternative language for certain types of work with fractions) and scientific notation (reading it and multiplying and dividing numbers written in this way).

These topics arise throughout applications of mathematics, and are an essential feature of answers to real-world word problems and questions in mathematical models (e.g., distances are never raw numbers; they are always some amount of kilometers, miles, etc.). Attention to units of measurement is necessary for meaningful answers to quantitative questions about the world.

8. Study trigonometry. Trigonometry admits different layers of understanding: the visual interpretation with right triangles (relating angles to lengths of sides based on similarity, including some special angles), the Law of Cosines for work with more general triangles, and the unit circle (explaining why sine and cosine relate to periodic phenomena and visualizing that $\sin^2 x + \cos^2 x = 1$).

The traditional blizzard of “trigonometric identities" is not truly important (though it gives opportunities for experience with proofs in an algebraic setting). In data science a measure of “closeness" of vectors is a reinterpretation of the Law of Cosines, as is the notion of correlation between two data sets, and anyone who will do college-level work in engineering, physical sciences, or math (e.g., calculus) needs exposure to trigonometry up through the unit circle. For instance, some students desire to pursue computer graphics, such as for video game design, and this cannot be done without a solid command of trigonometry.

9. Be able to apply logical reasoning and justification. Students should use careful arguments from hypotheses and definitions (and prior results) to arrive step-by-step at reliable conclusions. They also need experience critiquing the reasoning of others. Although traditionally done in the context of plane geometry, such justification can also be done with algebra (e.g., mathematical induction to establish some formulas). Some students are predisposed towards visualization and others towards formulas, so exposure to the idea of proof or justification via reasoning in both algebra and geometry makes principles more accessible.

What matters is practice with justifying steps in an argument, seeing logical reasoning used to arrive at results that are sometimes not evident (e.g., the Pythagorean Theorem, some facts about angles inscribed in circles, and the formula for $1 + 2 + 3 + ... + n$), identifying flaws in an incorrect argument (e.g., overlooking division by 0 that masks a counterexample, making an algebra error, etc.), and knowing the internal consistency of correct mathematical results (i.e., two correct facts in math are *never* incompatible). Diagnosing bugs in computer programs requires the capacity for clear thinking that is provided only by experience with this aspect of mathematics. Proofs and justifications should be seen not as a mechanical cookbook of rules to follow, but as a reliable means of arriving at correct conclusions and gaining understanding. To the extent possible, students should see some logical arguments where the conclusion is an unexpected or surprising result.

10. **Understand geometry in the plane, both visually and algebraically.** This includes understanding a variety of facts about polygons, angles, lines, circles, relations between similar triangles, the Pythagorean Theorem, and equations expressing circles and lines in algebraic form via coordinate geometry.

Geometric knowledge with similar triangles is further enhanced later on by using appropriate trigonometric functions to compute side lengths of a right triangle when given an angle (important for computing distances) and relating arc length along a circle to the angle of a sector. Vectors in the plane and transformations of a plane (rotation, dilation, shearing, etc.) provide a connection between algebra and geometry (with parallelograms and triangles) that is of great importance in data science (e.g., linear algebra) and physics (and in work with complex numbers).

11. **Be familiar with basic ideas from probability and statistics.** These ideas include independence of events, conditional probability, mean, variance, and learning from data. There is a vast array of applications of these ideas, illustrated by coin tosses, heredity, the difficulty of testing for rare diseases, the prosecutor's fallacy, and finding the best-fit line for planar data (and related concepts, such as correlation coefficient, slope, *y*-intercept, etc.). Both log-log plots and specific probability distributions (such as the normal distribution, binomial distribution, and Poisson distribution) with their precise

symbolic definitions extend and reinforce experience with exponentials, logarithms, and other concepts from algebra (as well as the notion of function).

12.* **Understand complex numbers.** This includes knowing how to add, subtract, multiply, and divide complex numbers (writing numbers in the standard form $a+bi$), and using complex numbers to solve a quadratic equation with real coefficients. The visual meaning of complex numbers is important for providing a concrete interpretation of them. When a student has learned trigonometry, the polar form $r\ (\cos(\vartheta) + i\ \sin(\vartheta))$ provides a valuable visual meaning for multiplication and reinforcement of some facts from trigonometry (e.g., addition laws for sine and cosine).

The topic of complex numbers extends experience with the universality of the laws of algebra and enhances mathematical maturity for appreciating the role of definitions in mathematics and learning broader conceptions of “algebra” later on (e.g., linear algebra in college, which pervades all quantitative modeling). The topic of complex numbers is fundamental for college-level work involving physics, chemistry, and engineering and is closely related to some topics in computer science (e.g., Google’s PageRank algorithm, the algebraic systems used in error-correcting codes and encryption, and the discrete Fourier transform in machine learning and data analysis).

* This item has an asterisk because it is of tremendous importance in some quantitative fields (chemistry, engineering, physics, math) but not others (e.g., biology and economics).