# GLOBAL CRYSTAL BASES AND $q$-SCHUR ALGEBRAS

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Abstract. We prove that the quantized Carter-Lusztig basis for a finite dimensional irreducible $U_q(\mathfrak{gl}_n(\mathbb{C}))$-module $V(\lambda)$ is related to the global crystal basis for $V(\lambda)$ by an upper triangular invertible matrix. We express the global crystal basis in terms of the $q$-Schur algebra and provide an algorithm for obtaining global crystal basis vectors for $V(\lambda)$ using the $q$-Schur algebra.

## 1. Introduction

Various bases for finite dimensional irreducible polynomial representations of the quantized universal enveloping algebra $U_q(\mathfrak{gl}_n(\mathbb{C}))$ have been given. Each such $U_q(\mathfrak{gl}_n)$-module is of the form $V(\lambda)$, where $\lambda$ is a partition of a positive integer into at most $n$ parts, and the dimension of $V(\lambda)$ is given by the number of semistandard $\lambda$-tableaux with entries in the set $\{1, 2, \ldots, n\}$. Several authors have studied transition matrices between various bases (see, for instance, [2], [4], [13]).

The canonical bases or global crystal bases of $V(\lambda)$ due to Lusztig [14] and Kashiwara [12] have nice properties but can be difficult to compute explicitly. Algorithms to compute global crystal basis vectors are given by de Graaf in [4] and Leclerc-Toffin in [13]. By embedding $V(\lambda)$ into a tensor product of fundamental modules, Leclerc and Toffin give an intermediate monomial basis for $V(\lambda)$ which is shown to be related to the global crystal basis of $V(\lambda)$ by a unitriangular matrix. They then obtain the global crystal basis vectors through a triangular algorithm.

Polynomial representations of $U_q(\mathfrak{gl}_n)$ can also be studied by means of the $q$-Schur algebra, $S_q(n, r)$. This is a quantized version of the classical Schur algebra $S(n, r)$ which was defined by J. A. Green [7] as the dual of the coalgebra $A(n, r)$ of homogeneous polynomials of degree $r$ in $n^2$ variables $x_{ij}$, $1 \leq i, j \leq n$. There are several different approaches to studying $q$-Schur algebras in the literature (see [1],[5],[6], [17]). We follow the approach taken by J. A. Green, but in the quantum setting (see [17]), where $A_q(n)$ is the coordinate ring of quantum matrices, due to Manin [15], $A_q(n, r)$ is the $r$th homogeneous part of $A_q(n)$, and $S_q(n, r)$ is the dual $A_q(n, r)^*$.

A quantized version of the Carter-Lusztig basis for $V(\lambda)$, given in terms of elements in $U_q(\mathfrak{gl}_n)^+$, is given in [18]. In [3], we give the Carter-Lusztig basis in

2010 *Mathematics Subject Classification.* 20G42, 20G43.

*Key words and phrases.* $q$-schur algebra, quantum group, global crystal basis, Carter-Lusztig basis, Young tableau.

This research was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

This article appears in *Journal of Algebra and its Applications*, **14** (8), Oct. 2015, 1 –21.

terms of $q$-Schur algebra elements. The primary aims of the current work are to describe the global crystal basis in terms of elements in the $q$-Schur algebra, to give an algorithm that explicitly provides elements of the global crystal basis using $q$-Schur algebra elements, and to prove that the Carter-Lusztig basis and global crystal basis are related by an invertible, upper triangular matrix.

After recalling the necessary background material, we discuss Leclerc-Toffin's intermediate basis in Section 4. We then develop various results regarding $q$-Schur algebras that allow us to explicitly prove at the end of Section 6 that the transition matrix between the quantized Carter-Lusztig basis and the Leclerc-Toffin intermediate basis is upper triangular and invertible, from which it follows that the Carter-Lusztig basis and global crystal basis are related by an invertible, upper triangular, matrix. We give a method for determining the entries of the first matrix in Section 7. This, combined with the algorithm for writing global basis vectors in terms of the intermediate basis elements allow us to give an algorithm for finding global basis vectors in terms of $q$-Schur algebra elements.

## 2. Young tableaux

Let $n$ and $r$ be fixed positive integers and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ and $\sum_{i=1}^{k} \lambda_i = r$, be a *partition* of $r$, denoted $\lambda \dashv r$. Define

$$\Lambda^+(n,r) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \dashv r \mid k \leq n\} \text{ and}$$

$$I(n,r) = \{I = (i_1, i_2, \ldots, i_r) \mid i_\rho \in \{1, \ldots n\},\ 1 \leq \rho \leq r\}.$$

All partitions $\lambda$ shall belong to $\Lambda^+(n,r)$. The *Young diagram of shape* $\lambda$ consists of $k$ left-justified rows where the $i$-th row contains $\lambda_i$ boxes and a $\lambda$-tableau is a filling of the Young diagram of shape $\lambda$ with entries from $\{1, 2, \ldots, n\}$.

A $\lambda$-tableau is *semistandard* if it is both column increasing and weakly row increasing. Denote the set of $\lambda$-tableau by $\mathcal{T}(\lambda, n)$ and let

$$CT(\lambda, n) = \{T \in \mathcal{T}(\lambda, n) \mid T \text{ is column increasing}\},$$

$$RT(\lambda, n) = \{T \in \mathcal{T}(\lambda, n) \mid T \text{ is weakly row increasing}\},$$

$$SST(\lambda, n) = \{T \in \mathcal{T}(\lambda, n) \mid T \text{ is semistandard}\}.$$

The *column sequence* $I_C(T)$ of $T$ comes from reading the entries down columns from left to right and the *row sequence* $I_R(T)$ from reading the entries across the rows of $T$ from top to bottom. If $I = I_R(T)$ is the row sequence of $T$, we will often write $I^t$ to denote the corresponding column sequence $I_C(T)$ of $T$.

We will often work with the column and row sequences of the tableau $T(\lambda)$, which is obtained by filling the $i$-th row of the Young diagram of shape $\lambda$ entirely with $i$'s. Denote $I_R(T(\lambda)) = I(\lambda)$ and $I_C(T(\lambda)) = I_C(\lambda)$.

The symmetric group acts on $I(n,r)$ by $I\sigma = (i_1, \ldots, i_r)\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(r)})$, for $\sigma \in S_r$, which yields an action on $\lambda$-tableaux by defining $T\sigma = S$ where $I_C(S) = I_C(T)\sigma$. Let $T^\lambda$ be the $\lambda$-tableau with row sequence $I_R(T^\lambda) = (1, 2, \ldots, r)$ and define $C(\lambda)$ to be the subgroup of permutations in $S_r$ that leave the columns of $T^\lambda$ invariant and $R(\lambda)$ the subgroup that leaves the rows of $T^\lambda$ invariant. Two $\lambda$-tableaux $T$ and $S$ are *row equivalent* if $T = S\sigma$ for some $\sigma \in R(\lambda)$; we denote this by $T \sim_R S$. Similarly, $T$ is *column equivalent* to $S$, written $T \sim_C S$, if $T = S\sigma$ where $\sigma \in C(\lambda)$.

**Example 2.1.** If $\lambda = (3,2,1)$ then $I(\lambda) = (1,1,1,2,2,3)$ and $I_C(\lambda) = (1,2,3,1,2,1)$.

For the semistandard $\lambda$-tableau $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$, we have $I_R(T) = (1,2,2,3,4,5)$,

$I_C(T) = (1,3,5,2,4,2)$ and $\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \sim_R \begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline 4 & 3 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$.

## 3. Quantized enveloping algebras and $q$-Schur algebras

Let $q$ be an indeterminate. The *quantized enveloping algebra* of the complex Lie algebra $\mathfrak{gl}_n$, denoted $U_q(\mathfrak{gl}_n)$, is the associative algebra over $\mathbb{C}(q)$ with generators $E_i, F_i$, $1 \le i < n$, $K_i, K_i^{-1}$, $1 \le i \le n$ and relations as follows:

$$K_iK_i^{-1} = K_i^{-1}K_i = 1 \qquad K_iK_j = K_jK_i$$
$$K_iE_j = q^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i \qquad K_iF_j = q^{\delta_{i,j+1}-\delta_{i,j}}F_jK_i$$
$$E_iE_j = E_jE_i \text{ if } |i-j| > 1 \qquad F_iF_j = F_jF_i \text{ if } |i-j| > 1$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_{i,i+1} - K_{i,i+1}^{-1}}{q - q^{-1}}$$
$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \text{ if } |i-j| = 1$$
$$F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \text{ if } |i-j| = 1,$$

where $K_{i,i+1} = K_iK_{i+1}^{-1}$. The subalgebra of $U_q(\mathfrak{gl}_n)$ generated by all $E_i$, $1 \le i < n$ is denoted $U_q(\mathfrak{gl}_n)^+$ and the subalgebra generated by all $F_i$ is denoted by $U_q(\mathfrak{gl}_n)^-$.

The *natural module* is the $\mathbb{C}(q)$-vector space $V$ with basis $\{v_1, \dots, v_n\}$ and $U_q(\mathfrak{gl}_n)$-action given by $E_iv_k = \delta_{i+1,k}v_i, F_iv_k = \delta_{i,k}v_{i+1}, K_iv_k = q^{\delta_{i,k}}v_k$. This action can be extended to $V^{\otimes r}$ via the comultiplication $\Delta$ on $U_q(\mathfrak{gl}_n)$ defined by

(1) $\Delta(E_i) = E_i \otimes 1 + K_{i,i+1}^{-1} \otimes E_i,\ \ \Delta(F_i) = F_i \otimes K_{i,i+1} + 1 \otimes F_i,\ \ \Delta(K_j) = K_j \otimes K_j,$

$1 \le i < n,\ 1 \le j \le n.$

Let $\tau : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n)$ be the antiautomorphism given by

$$\tau(E_i) = F_i, \quad \tau(F_i) = E_i, \quad \tau(K_j) = K_j,\ 1 \le i < n,\ 1 \le j \le n.$$

Another comultiplication $\Delta_1 : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$ is given by $\Delta_1 = \tau \otimes \tau \circ \Delta \circ \tau$. We then have the following:

(2) $\Delta_1(E_i) = 1 \otimes E_i + E_i \otimes K_{i,i+1}, \Delta_1(F_i) = K_{i,i+1}^{-1} \otimes F_i + F_i \otimes 1, \Delta_1(K_j) = K_j \otimes K_j,$

where $1 \le i < n,\ 1 \le j \le n.$

**Remark 1.** We will need to make minor adjustments to some of the required results from [3] and [11] since the comultiplication $\Delta_1$ was used in those articles.

For a $U_q(\mathfrak{gl}_n)$-module $V$ and $\chi = (\chi_1, \dots, \chi_n)$ an $n$-tuple of non-negative integers, the *weight space* associated to $\chi$ is the subspace $V^\chi = \{v \in V \mid K_i v = q^{\chi_i} v,\ 1 \le i \le n\}$. If $v \in V^\chi$, $v \neq 0$, then $v$ is said to be a *weight vector* of weight $\chi$, and $v$ is a *highest-weight vector* if $E_i v = 0$ for $1 \le i < n$.

Let $A_q(n)$ be the associative $\mathbb{C}(q)$-algebra generated by the variables $x_{ij}$, $1 \le i, j \le n$, subject to the relations (see [15], [19], for instance):

$$\begin{array}{ll} x_{il}x_{ik} = qx_{ik}x_{il} & 1 \le k < l \le n \\ x_{jk}x_{ik} = qx_{ik}x_{jk} & 1 \le i < j \le n \\ x_{il}x_{jk} = x_{jk}x_{il} & 1 \le i < j \le n, 1 \le k < l \le n \\ x_{ik}x_{jl} - x_{jl}x_{ik} = (q^{-1} - q)x_{il}x_{jk} & 1 \le i < j \le n,\ 1 \le k < l \le n. \end{array} \tag{3}$$

Given $I = (i_1, \dots, i_r),\ J = (j_1, \dots, j_r) \in I(n, r)$, let $x_{I,J} = x_{i_1 j_1} \cdots x_{i_r j_r} \in A_q(n)$ and let $A_q(n, r)$ denote the $\mathbb{C}(q)$-subspace of $A_q(n)$ generated by the monomials $x_{I,J}$, where $I, J \in I(n, r)$. The algebra $A_q(n)$ is a coalgebra, with comultiplication given by $\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj}$, and $A_q(n, r)$ is a subcoalgebra of $A_q(n)$. The dual $A_q(n, r)^* = S_q(n, r)$, is then an associative $\mathbb{C}(q)$-algebra called the *q-Schur algebra* with multiplication $\xi\eta(x_{I,J}) = \sum_{A \in I(n,r)} \xi(x_{I,A})\eta(x_{A,J})$, where $\xi, \eta \in S_q(n, r),\ x_{I,J} \in A_q(n, r)$.

Let $I(n, r)^2 = I(n, r) \times I(n, r)$ and define

$$\mathcal{J}(n, r) = \{(I, J) \in I(n, r)^2 \mid j_1 \le j_2 \le \cdots \le j_r \text{ and } i_k \le i_{k+1} \text{ when } j_k = j_{k+1}\}.$$

Then $\{x_{I,J} \mid (I, J) \in \mathcal{J}(n, r)\}$ is a basis for $A_q(n, r)$ (see [5]). We will often shorten the notation for $\mathcal{J}(n, r)$ to $\mathcal{J}$.

The dual basis $\{\xi_{I,J} \mid (I, J) \in \mathcal{J}(n, r)\}$ for $S_q(n, r)$ satisfies $\xi_{I,J}(x_{P,Q}) = 1$ if $x_{P,Q} = x_{I,J}$ and $\xi_{I,J}(x_{P,Q}) = 0$ otherwise, where $(P, Q), (I, J) \in \mathcal{J}(n, r)$. For arbitrary $(I, J) \in I(n, r)^2$, we define

$$\xi_{I,J} = \sum_{(A,B) \in \mathcal{J}} c_{A,B}\xi_{A,B} \quad \text{where } x_{I,J} = \sum_{(A,B) \in \mathcal{J}} c_{A,B}x_{A,B}.$$

The symmetric group acts on $I(n, r) \times I(n, r)$ by $(I, J)\sigma = (I\sigma, J\sigma)$. Let $<$ be the lexicographic order on $I(n, r)$ and order $I(n, r) \times I(n, r)$ by defining $(A, B) < (I, J)$ if $B < J$ or $B = J$ and $A < I$. Let $(I, J)_0$ be the minimal element in the $S_r$-orbit containing $(I, J)$.

For $I = (i_1, i_2, \dots, i_r),\ J = (j_1, j_2, \dots, j_r) \in I(n, r)$, let $S_1 = \{(a, b) \mid a < b,\ i_a = i_b \text{ and } j_a > j_b\}$, $S_2 = \{(a, b) \mid a < b,\ j_a = j_b \text{ and } i_a > i_b\}$, and define $\epsilon(I, J) = |S_1| + |S_2|$. The following two lemmas, the first of which is an adjustment of [16, Lemma 6.1.2], will be useful throughout the article.

**Lemma 3.1.** *Let $I, J \in I(n, r)$. Then* $x_{I,J} = q^{\varepsilon(I,J)}x_{(I,J)_0} + \sum_{\substack{(S,T) \in \mathcal{J} \\ (S,T) > (I,J)_0}} a_{S,T}x_{S,T}$,

*where $a_{S,T} \in \mathbb{Z}[q, q^{-1}]$.*

Define $\mathcal{R}(\lambda, n) = \{Q \in I(n, r) \mid Q = I_R(T) \text{ for some } T \in RT(\lambda, n)\} = \{Q \in I(n, r) \mid (Q, I(\lambda)) \in \mathcal{J}\}$.

**Lemma 3.2.** *Let $\eta \in S_q(n,r)$ and $\lambda \in \Lambda^+(n,r)$. Then* $\eta\xi_{I(\lambda),I(\lambda)} = \sum_{Q\in\mathcal{R}(\lambda,n)} a_Q \xi_{Q,I(\lambda)}$, *where $a_Q \in \mathbb{C}(q)$.*

*Proof.* Write $\eta\xi_{I(\lambda),I(\lambda)} = \sum_{(Q,P)\in\mathcal{J}} a_{Q,P}\xi_{Q,P}$ as a $\mathbb{C}(q)$-linear combination of basis elements. Then $a_{Q,P} = \eta\xi_{I(\lambda),I(\lambda)}(x_{Q,P}) = \sum_{A\in I(n,r)} \eta(x_{Q,A})\xi_{I(\lambda),I(\lambda)}(x_{A,P})$. But $\xi_{I(\lambda),I(\lambda)}(x_{A,P}) = 0$ unless $P \sim I(\lambda)$ and since $(Q,P) \in \mathcal{J}$, we must have $P = I(\lambda)$. Thus $\eta\xi_{I(\lambda),I(\lambda)} = \sum_{(Q,I(\lambda))\in\mathcal{J}} a_Q\xi_{Q,I(\lambda)}$, and $(Q, I(\lambda)) \in \mathcal{J}$ if and only if the tableau with row sequence $Q$ is weakly row increasing. □

## 4. Leclerc-Toffin bases and global crystal bases

We review the relevant results on $U_q(\mathfrak{gl}_n)$-modules and global bases, for the most part following [13]. We have a $U_q(\mathfrak{gl}_n)$-module action on $A_q(n)$ given by

$$E_i x_{kl} = \delta_{i+1,l}x_{ki}, \quad F_i x_{kl} = \delta_{il}x_{k,i+1}, \quad K_i x_{kl} = q^{\delta_{il}}x_{kl}, \quad K_i^{-1}x_{kl} = q^{-\delta_{il}}x_{kl} \tag{4}$$

and, using (1), we have

$$E_i(PQ) = (E_iP)Q + (K_{i,i+1}^{-1}P)(E_iQ), \;\; F_i(PQ) = (F_iP)(K_{i,i+1}Q) + P(F_iQ),$$
$$K_i(PQ) = (K_iP)(K_iQ), \quad P, Q \in A_q(n).$$

Given $I = (i_1, i_2, \ldots, i_r)$, $J = (j_1, j_2, \ldots, j_r) \in I(n,r)$ with $i_1 < i_2 < \cdots < i_r$, define the *$q$-determinant* in $A_q(n,r)$ by

$$\det{}_q X_J^I = \begin{cases} \sum_{\sigma\in S_r}(-q)^{-\ell(\sigma)}x_{i_1 j_{\sigma(1)}}x_{i_2 j_{\sigma(2)}}\cdots x_{i_r j_{\sigma(r)}} & \text{if } j_1 < j_2 < \cdots < j_r \\ \sum_{\sigma\in S_r}(-q)^{-\ell(\sigma)}x_{i_{\sigma(1)} j_1}x_{i_{\sigma(2)} j_2}\cdots x_{i_{\sigma(r)} j_r} & \text{otherwise.} \end{cases}$$

For $k \le n$, let $\Lambda_k = (\underbrace{1,1,\ldots,1}_{k}, 0, \ldots, 0)$ and let $T$ be a $\Lambda_k$-tableau with column sequence $I_C(T) = (a_1, a_2, \ldots, a_k)$ where $a_i \in \{1, \ldots, n\}$ for $1 \le i \le k$. Associate to $T$ an element $\omega(T) \in A_q(n,r)$, called a (one-column) *bideterminant* by

$$\omega(T) = \det{}_q X^{1,2,\ldots,k}_{a_1,a_2,\ldots,a_k}.$$

The following lemma follows from the relations (3).

**Lemma 4.1.** *Let $T$ be a one-column $\Lambda_k$-tableau. Then*

(1) *$\omega(T) = 0$ if $T$ contains repeated entries and*
(2) *if $T$ is column increasing and $T = S\sigma$ then $\omega(T) = (-q)^{\ell(\sigma)}\omega(S)$.*

The $\mathbb{C}(q)$-vector space generated by one-column bideterminants $\omega(T)$ given by $\Lambda_k$-tableaux is a $U_q(\mathfrak{gl}_n)$-module, called a *fundamental module*, with action given by (4); we denote this $U_q(\mathfrak{gl}_n)$-module by $V(\Lambda_k)$. We have the following lemma, which follows readily by use of the relations (3).

**Lemma 4.2.** *Let $T$ be a one-column $\Lambda_k$-tableau with $\omega(T) \neq 0$.*

(1) *If $T$ contains an $i+1$, then $E_i\omega(T)=\omega(S)$ where $S$ is the same as $T$ except that the $i+1$ has been replaced with an $i$. If $T$ does not contain an $i+1$, then $E_i\omega(T)=0$.*
(2) *If $T$ contains an $i$, then $F_i\omega(T)=\omega(S)$ where $S$ is the same as $T$ except that the $i$ has been replaced by an $i+1$. If $T$ does not contain an $i$, then $F_i\omega(T)=0$.*
(3) *If $T$ contains an $i$, then $K_i\omega(T)=q\omega(T)$ and $K_i\omega(T)=\omega(T)$ otherwise.*

Let $\lambda=\sum_{i=1}^{n}a_i\Lambda_i\in\Lambda^+(n,r)$ and let

$$W(\lambda)=V(\Lambda_n)^{\otimes a_n}\otimes V(\Lambda_{n-1})^{\otimes a_{n-1}}\otimes\cdots\otimes V(\Lambda_1)^{\otimes a_1}.$$

A basis for $W(\lambda)$ is given by

$$\mathcal{B}_W(\lambda)=\{\omega(T)\mid T\in\mathrm{CT}(\lambda,n)\}.$$

Define $w_\lambda\in W(\lambda)$ to be the tensor product of the highest-weight vectors of each $V(\Lambda_k)$. Then $w_\lambda$ has weight $\lambda$ and is the unique highest-weight vector (up to scalars) in $W(\lambda)$. The $U_q(\mathfrak{gl}_n)$-module $V(\lambda)=U_q(\mathfrak{gl}_n)w_\lambda$ is irreducible and every finite dimensional irreducible polynomial $U_q(\mathfrak{gl}_n)$-module is isomorphic to some $V(\lambda)$ where $\lambda\in\Lambda^+(n,r)$. A basis for $V(\lambda)$ is indexed by the elements $T\in SST(\lambda,n)$ (see, for instance, [9]).

The *canonical basis* or (lower) *global basis* for $U_q(\mathfrak{g})^-$, where $\mathfrak{g}$ is a complex simple Lie algebra, was first introduced by Lusztig in [14]. Another proof of the existence of canonical bases was later given by Kashiwara in [12]. The canonical bases induce bases for $V(\lambda)$. For a general introduction to crystal bases, see [8] or [10]. Following [13], we recall the definition of the global crystal basis of a $U_q(\mathfrak{gl}_n)$-module $V(\lambda)$.

Let $\mathcal{A}$ be the subring of $\mathbb{C}(q)$ of rational functions without pole at $q=0$. Let $L_W(\lambda)$ denote the $\mathcal{A}$-lattice in $W(\lambda)$ spanned by the basis elements in $\mathcal{B}_W(\lambda)$, which is the *crystal lattice* of $W(\lambda)$. Let $L_V(\lambda)=L_W(\lambda)\cap V(\lambda)$, which is the crystal lattice of $V(\lambda)$.

Define a $\mathbb{C}(q)$-algebra homomorphism on $U_q(\mathfrak{gl}_n)$ that is an involution by

$$\overline{E_i}=E_i,\quad \overline{F_i}=F_i,\quad \overline{q}=q^{-1},\quad \overline{K_j}=K_j^{-1},\qquad 1\le i<n,\ 1\le j\le n, \tag{5}$$

and define $\overline{w}=\overline{u}w_\lambda$, where $w=uw_\lambda$ for $u\in U_q(\mathfrak{gl}_n)$.

Let $U_{\mathbb{Q}}^-$ denote the $U_q(\mathfrak{gl}_n)$-subalgebra generated over $\mathbb{Q}[q,q^{-1}]$ by the divided powers $F_i^{(k)}:=\dfrac{F_i^k}{[k]!}$, where $[k]!=[k][k-1]\cdots[1]$ and $[m]=\dfrac{q^m-q^{-m}}{q-q^{-1}}$, and let $V_{\mathbb{Q}}(\lambda)=U_{\mathbb{Q}}^-w_\lambda$. We have the following theorem (see [14] and [12]).

**Theorem 4.3.** *There exists a unique $\mathbb{Q}[q,q^{-1}]$-basis $\{G(T)\mid T\in SST(\lambda,n)\}$ of $V_{\mathbb{Q}}(\lambda)$ with the properties that*
(1) $G(T)\equiv\omega(T)$ *mod* $qL_W(\lambda)$,
(2) $\overline{G(T)}=G(T)$.

This basis is called the *global crystal basis* of $V(\lambda)$.

We now recall the monomial basis for $V(\lambda)$ which was introduced in [13]. Given a semistandard $\lambda$-tableau $T$, let $i$ be the smallest integer such that $i+1$ appears in

$T$ in a row with row number less than $i+1$. Let $r_1$ be the number of occurrences of $i+1$ that appear in any row with row number less than $i+1$ and let $i_1 = i$. Form a new $\lambda$-tableau $T_1$ by replacing the $r_1$ occurrences of $i+1$ by $i$. Repeat the procedure with $T_1$ to give integers $r_2$ and $i_2$ and a tableau $T_2$. After the procedure terminates to give $T(\lambda)$, we obtain two sequences $(i_1, i_2, \ldots, i_s)$ and $(r_1, r_2, \ldots, r_s)$. Define $a(T) = F_{i_1}^{(r_1)} \cdots F_{i_s}^{(r_s)} \in U_q(\mathfrak{gl}_n)^-$.

**Example 4.4.** If $T = \begin{array}{|c|c|c|c|c|}\hline 1&2&2&3&4\\ \hline 3&4&4 \\ \cline{1-3}\end{array}$ then $a(T) = F_1^{(2)} F_2^{(2)} F_1 F_3^{(3)} F_2^{(3)} F_1$.

Given two column increasing $\lambda$-tableaux $S$ and $T$, let $S < T$ if $I_C(S) < I_C(T)$. Lemmas 4.5–4.6 and Theorem 4.7 are proved in [13].

**Lemma 4.5.** *Let $T \in SST(\lambda, n)$ and suppose that $a(T)w_\lambda = \sum_{S \in CT(\lambda,n)} \alpha_{ST}(q)\omega(S)$ as a linear combination of basis elements in $\mathcal{B}_W(\lambda)$. Then $\alpha_{ST}(q) \in \mathbb{N}[q, q^{-1}]$, $\alpha_{TT} = 1$ and $\alpha_{ST}(q) \neq 0$ only if $S \geq T$. Furthermore, $\alpha_{ST}(q) = 0$ unless $\omega(S)$ and $\omega(T)$ have the same weight.*

It follows from the above lemma that $\{a(T)w_\lambda \mid T \in SST(\lambda, n)\}$ is a basis for $V(\lambda)$. In the lemma and theorem below, let $\{G(T) \mid T \in SST(\lambda, n)\}$ be the global basis for $V(\lambda)$.

**Lemma 4.6.** *Let $T \in SST(\lambda, n)$ and suppose that the expansion of $G(T)$ in the basis $\{a(T)w_\lambda \mid T \in SST(\lambda, n)\}$ is $G(T) = \sum_{S \in SST(\lambda,n)} \beta_{ST}(q)a(S)w_\lambda$. Then $\beta_{TT}(q) = 1$, and $\beta_{ST}(q) = 0$ unless $S \geq T$.*

**Theorem 4.7.** *Let $T \in SST(\lambda, n)$ and suppose that $G(T) = \sum_{S \in CT(\lambda,n)} d_{ST}(q)\omega(S)$ as a linear combination of basis elements in $\mathcal{B}_W(\lambda)$. Then*

1. *$d_{ST}(q) \in \mathbb{Z}[q]$,*
2. *$d_{TT}(q) = 1$ and $d_{ST}(0) = 0$ if $S \neq T$,*
3. *$d_{ST}(q) = 0$ unless $\omega(S)$ and $\omega(T)$ have the same weight and $S \geq T$.*

Using the above, one can obtain the global crystal basis $\{G(T) \mid T \in SST(\lambda, n)\}$ by a triangular algorithm. Let $T^{(1)}, T^{(2)}, \cdots, T^{(t)}$ be the tableaux in $SST(\lambda, n)$ numbered such that $T(\lambda) = T^{(1)} < T^{(2)} < \cdots < T^{(t)}$. Certainly $G(T^{(t)}) = a(T^{(t)})w_\lambda$ and, $G(T^{(t-1)}) = a(T^{(t-1)})w_\lambda - \gamma_t(q)G(T^{(t)})$, where $\gamma_t(q) \in \mathbb{Q}[q, q^{-1}]$. Since $\overline{G(T^{(i)})} = G(T^{(i)})$ for $1 \leq i \leq t$, $\gamma_t(q) = \gamma_t(q^{-1})$. Furthermore, $G(T^{(t-1)}) \equiv \omega(T^{(t-1)}) \mod qL_W(\lambda)$, so writing $a(T^{(t-1)})w_\lambda - \gamma_t(q)G(T^{(t)})$ as a linear combination of basis elements in $\mathcal{B}_W(\lambda)$ and using these two facts determines $\gamma_t(q)$.

More generally, if one has written each of $G(T^{(i+1)}), G(T^{(i+2)}), \ldots, G(T^{(t)})$ as a linear combination of basis vectors in $\mathcal{B}_W(\lambda)$, then the coefficients in the linear combination $G(T^{(i)}) = a(T^{(i)})w_\lambda - \gamma_{i+1}(q)G(T^{(i+1)}) - \cdots - \gamma_t(q)G(T^{(t)})$ are completely determined by the facts that

$$\gamma_k(q^{-1}) = \gamma_k(q),\ 1 \leq k \leq t,\ G(T^{(i)}) \equiv \omega(T^{(i)}) \mod qL_W(\lambda).$$

For an example, see [13] or Example 7.5.

## 5. Carter-Lusztig Bases and $q$-Schur algebras

In [18], a quantum version of the Carter-Lusztig basis of the $q$-Weyl module, which is isomorphic to $V(\lambda)$ as a $U_q(\mathfrak{gl}_n)$-module, is given using elements in $U_q(\mathfrak{gl}_n)^-$. In [3], it is shown that the elements in the Carter-Lusztig basis can be written in terms of elements in the $q$-Schur algebra up to a power of $q$. Since the $q$-Schur algebra version of this basis is easier to work with than the $U_q(\mathfrak{gl}_n)$ version, we use it to prove that this basis is related to the Leclerc-Toffin basis by an upper triangular matrix and provide a method for writing elements in the Leclerc-Toffin basis using elements in the $q$-Schur algebra. We then adjust the Leclerc-Toffin algorithm to obtain the global basis for $V(\lambda)$ in terms of elements in the $q$-Schur algebra. We first recall the construction of the quantum Carter-Lusztig basis.

Define $F_{i,i+1} = F_i$ and for $|i-j| \geq 1$ define $F_{ij},\ E_{ij} \in U_q(\mathfrak{gl}_n)$ recursively as

$$F_{ij} = F_{i+1,j}F_i - q^{-1}F_iF_{i+1,j}, \quad E_{ij} = E_iE_{i+1,j} - q^{-1}E_{i+1,j}E_i.$$

For a semistandard $\lambda$-tableau $T$ with $k \leq n$ rows, define $F_T,\ E_T \in U_q(\mathfrak{gl}_n)$ by

$$F_T = \prod_{1\leq i<k,\ i<j\leq n} F_{ij}^{(\gamma_{ij})} = F_{12}^{(\gamma_{12})}F_{13}^{(\gamma_{13})}\cdots F_{1k}^{(\gamma_{1k})}F_{23}^{(\gamma_{23})}\cdots F_{2k}^{(\gamma_{2k})}\cdots F_{k-1,k}^{(\gamma_{k-1,k})},$$

$$E_T = \prod_{1\leq i<k,\ i<j\leq n} E_{ij}^{(\gamma_{ij})} = E_{k-1,k}^{(\gamma_{k-1,k})}\cdots E_{2k}^{(\gamma_{2k})}\cdots E_{23}^{(\gamma_{23})}E_{1k}^{(\gamma_{1k})}\cdots E_{13}^{(\gamma_{13})}E_{12}^{(\gamma_{12})},$$

where $\gamma_{ij}$ is the number of $j$'s in row $i$ of $T$, and $k$ is the number of columns in $T$.

For $I = (i_1, i_2, \ldots, i_r) \in I(n,r)$, let $v_I = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \in V^{\otimes r}$. Define a bilinear form $\langle\ ,\ \rangle : V^{\otimes r} \times V^{\otimes r} \to \mathbb{Q}[q,q^{-1}]$ by $\langle v_I, v_J\rangle = \delta_{I,J}$. The following Lemma reveals the relationship between the two comultiplications $\Delta$ and $\Delta_1$.

**Lemma 5.1.** *Let $u \in U_q(\mathfrak{gl}_n),\ v, w \in V^{\otimes r}$. Then $\langle \Delta^{r-1}(u)v, w\rangle = \langle v, \Delta_1^{r-1}(\tau(u))w\rangle$.*

*Proof.* It suffices to prove that $\langle \Delta^{r-1}(F_i)v_I, v_J\rangle = \langle v_I, \Delta_1^{r-1}(E_i)v_J\rangle$, where $1 \leq i < n$ and $I = (i_1, i_2, \ldots, i_r),\ J = (j_1, j_2, \ldots, j_r) \in I(n,r)$.

We have

$\Delta^{r-1}(F_i)v_I = v_{i_1}\otimes\cdots v_{i_{r-1}}\otimes(F_iv_{i_r})+\cdots+(F_iv_{i_1})\otimes(K_{i,i+1}v_{i_2})\otimes\cdots\otimes(K_{i,i+1}v_{i_r})$ and

$\Delta_1^{r-1}(E_i)v_J = v_{j_1}\otimes\cdots v_{j_{r-1}}\otimes(E_iv_{j_r})+\cdots+(E_iv_{j_1})\otimes(K_{i,i+1}v_{j_2})\otimes\cdots\otimes(K_{i,i+1}v_{j_r})$.

Since $\langle v_{i_1} \otimes \cdots \otimes (F_iv_{i_k}) \otimes \cdots \otimes (K_{i,i+1}v_{i_{r-1}}) \otimes (K_{i,i+1}v_{i_r}), v_J\rangle$ is the same as $\langle v_I, v_{j_1} \otimes \cdots \otimes (E_iv_{j_k}) \otimes \cdots \otimes (K_{i,i+1}v_{j_{r-1}}) \otimes (K_{i,i+1}v_{j_r})\rangle$, for $1 \leq k \leq r$, the result follows. □

Note that in the proofs below we will simply write $uv$ instead of $\Delta^{r-1}(u)v$ for $u \in U_q(\mathfrak{gl}_n)$ and $v \in V^{\otimes r}$ but when we are using the action of $V^{\otimes r}$ given by the comultiplication $\Delta_1$, this will always be specified.

Given $I = (i_1, i_2, \ldots, i_r) \in I(n,r)$, let $\beta(I) = |\{(a,b) \mid a < b \text{ and } i_a \neq i_b\}|$. From [18], we have both the following identity and Theorem 5.2:

$$q^{\beta(J)}\langle \Delta_1^{r-1}(u)v_I, v_J\rangle = q^{\beta(I)}\langle v_I, \Delta_1^{r-1}(\tau(u))v_J\rangle. \tag{6}$$

**Theorem 5.2.** *The set $\{F_Tw_\lambda \mid T \in SST(\lambda,n)\}$ is a basis for $V(\lambda)$.*

*Proof.* In [18], it is proved that $\{\Delta_1^{r-1}(F_T)z_\lambda \mid T \in SST(\lambda,n)\}$ is a basis for the $q$-Weyl module, $\Delta_q(\lambda)$, which is the $U_q(\mathfrak{gl}_n)$-submodule of $V^{\otimes r}$, generated by the highest-weight vector $z_\lambda = \sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} v_{I_C(\lambda)\sigma} \in V^{\otimes r}$. For a given $T \in SST(\lambda,n)$ with $I_R(T) = J$, write $F_T z_\lambda = \sum_{K \in I(n,r)} a_K v_K$, as a linear combination of basis elements in $V^{\otimes r}$. Since each $K$ in the sum has $K = J\sigma$ for some $\sigma \in S_r$, $\beta(K) = \beta(J)$. Furthermore, $a_K = \langle F_T z_\lambda, v_K\rangle = \langle z_\lambda, \Delta_1^{r-1}(E_T)z_\lambda\rangle$.

Since $\beta(I_C(\lambda)\sigma) = \beta(I(\lambda))$ for $\sigma \in C(\lambda)$, for each $K$ we have

$$\langle z_\lambda, \Delta_1^{r-1}(E_T)v_K\rangle = q^{\beta(K)-\beta(I(\lambda))}\langle \Delta_1^{r-1}(F_T)z_\lambda, v_K\rangle = q^{\beta(J)-\beta(I(\lambda))}\langle \Delta_1^{r-1}(F_T)z_\lambda, v_K\rangle.$$

It follows that $F_T z_\lambda = q^{\beta(J)-\beta(I(\lambda))}\Delta_1^{r-1}(F_T)z_\lambda$ so that $\{F_T z_\lambda \mid T \in SST(\lambda,n)\}$ is a basis for $\Delta_q(\lambda)$. Since the highest weight module $\Delta_q(\lambda)$ is isomorphic to $V(\lambda)$, the theorem now follows. □

Write $T_I$ for the tableau $T \in CT(\lambda,n)$ with column sequence $I$. Then $W(\lambda)$ is an $S_q(n,r)$-module, with action $\xi\omega(T_I) = \sum_{A \in I(n,r)} \xi(x_{A,I})\omega(T_A)$.

Given $v_J \in V^{\otimes r}$ and $u \in U_q(\mathfrak{gl}_n)$, define $\theta : U_q(\mathfrak{gl}_n) \to S_q(n,r)$ by $\theta(u)(x_{I,J}) = \langle uv_J, v_I\rangle$. The following lemma is proved in [11, Lemma 5.1, 5.2].

**Lemma 5.3.** *Let $\theta : U_q(\mathfrak{gl}_n) \to S_q(n,r)$ be as defined above, let $u, w \in U_q(\mathfrak{gl}_n)$ and $T \in CT(\lambda,n)$. Then*

(1) *$\theta(uw) = \theta(u)\theta(w)$ and*
(2) *$\theta(u)\omega(T) = u\omega(T)$.*

Define $\begin{pmatrix} K_i \\ t \end{pmatrix} = \prod_{s=1}^{t} \frac{q^{-s+1}K_i - q^{s-1}K_i^{-1}}{q^s - q^{-s}} \in U_q(\mathfrak{gl}_n)$, for $1 \le i, t \le n$. Suppose that $\lambda = (\lambda_1, \dots, \lambda_k)$, let $u_i = \begin{pmatrix} K_i \\ \lambda_i \end{pmatrix}$ and define $u^\lambda = \prod_{i=1}^{k} u_i \in U_q(\mathfrak{gl}_n)$.

**Lemma 5.4.** *For each $\lambda \in \Lambda^+(n,r)$, we have $\theta(u^\lambda) = \xi_{I(\lambda),I(\lambda)}$.*

*Proof.* We will prove that $\theta(u^\lambda) = \xi_{I(\lambda),I(\lambda)}$ by showing that $u^\lambda v_{I(\lambda)\sigma} = v_{I(\lambda)\sigma}$ for $\sigma \in S_r$ and that $u^\lambda v_J = 0$ for $J \in I(n,r)$ when $J \ne I(\lambda)\sigma$ for any $\sigma \in S_r$. Since $\prod_{s=1}^{\lambda_i} \frac{q^{-s+1+\lambda_i} - q^{s-1-\lambda_i}}{q^s - q^{-s}} = 1$, we have

$$\begin{aligned} u^\lambda v_{I(\lambda)\sigma} &= \prod_{i=1}^{k}\prod_{s=1}^{\lambda_i} \frac{q^{-s+1}K_i - q^{s-1}K_i^{-1}}{q^s - q^{-s}} v_{I(\lambda)\sigma} \\ &= \prod_{i=1}^{k}\prod_{s=1}^{\lambda_i} \frac{q^{-s+1}q^{\lambda_i} - q^{s-1}q^{-\lambda_i}}{q^s - q^{-s}}\, v_{I(\lambda)\sigma} = v_{I(\lambda)\sigma}. \end{aligned}$$

Consider $J \in I(n,r)$, with $J \ne I(\lambda)\sigma$ for any $\sigma \in S_r$. There must be some $m$ with $1 \le m \le k$ that appears $a_m$ times in the $r$-tuple $J$ with $a_m < \lambda_m$; let

$m$ be maximal with this property. Then $u^\lambda v_J = \prod_{i=1}^{k} u_i v_J = \prod_{i=1}^{m} u_i(\alpha(q)v_J)$, where $\alpha(q) \in \mathbb{Q}[q, q^{-1}]$ and

$$u_m v_J = \binom{K_m}{\lambda_m} v_J = \prod_{s=1}^{a_m} \frac{q^{-s+1}K_m - q^{s-1}K_m^{-1}}{q^s - q^{-s}} \frac{q^{-a_m}K_m - q^{a_m}K_m^{-1}}{q^{a_m+1} - q^{-(a_m+1)}} \ \beta(q)v_J,$$

where $\beta(q) \in \mathbb{Q}[q, q^{-1}]$. But $(q^{-a_m}K_m - q^{a_m}K_m^{-1})v_J = (q^{-a_m}q^{a_m} - q^{a_m}q^{-a_m})v_J = 0$, so that $u^\lambda v_J = 0$. $\square$

Let $T \in SST(\lambda, n)$. Denote the entry in the $i$-th row and $j$-th column of $T$ by $T_{ij}$ and define $s(T) = |\{(i, j, a, b) \mid i > j,\ a < b,\ T_{ia} = T_{jb}\}|$. By definition, $s(T)$ counts the number of pairs $(ia, jb)$ for which $T_{ia} = T_{jb}$ and $T_{ia}$ sits in a row below $T_{jb}$ and in a column to the left of $T_{jb}$. Define $r(T) = |\{(i, a, b) \mid a < b \leq \lambda_i,\ T_{ia} \neq T_{ib}\}|$.

The following theorem is an adjusted version of [3, Theorems 18, 19].

**Theorem 5.5.** *Let $T \in SST(\lambda, n)$ with $J = I_R(T)$. Then*

(1) $\theta(F_T)\xi_{I(\lambda),I(\lambda)} = q^{-s(T)}\xi_{J,I(\lambda)}$
(2) $\xi_{I(\lambda),I(\lambda)}\theta(E_T) = q^{-r(T)}\xi_{I(\lambda),J}$.

*Proof.* In [3] it was proved that $\langle v_{I(\lambda)}, \xi_{I(\lambda),I(\lambda)}\Delta_1^{r-1}(E_T)v_K\rangle = 0$ unless $K = J$ and that $\langle v_{I(\lambda)}, \Delta_1^{r-1}(E_T)\xi_{I(\lambda),I(\lambda)}v_J\rangle = q^{-s(T)}$. By Lemma 3.2, $\theta(F_T)\xi_{I(\lambda),I(\lambda)} = \theta(F_T u^\lambda) = \sum_{Q\in\mathcal{R}(\lambda,n)} a_Q \xi_{Q,I(\lambda)}$. Since

$$a_Q = \langle F_T u^\lambda v_{I(\lambda)}, v_Q\rangle = \langle v_{I(\lambda)}, \Delta_1^{r-1}(u^\lambda E_T)v_Q\rangle = \langle v_{I(\lambda)}, \xi_{I(\lambda),I(\lambda)}\Delta_1^{r-1}(E_T)v_Q\rangle,$$

we have $a_Q = 0$ unless $Q = J$ and $a_J = q^{-s(T)}$.

It was also proved in [3] that $\langle v_{I(\lambda)}, \xi_{I(\lambda),I(\lambda)}\Delta_1^{r-1}(F_T)v_K\rangle = 0$ unless $Q = J$ and that $\langle v_{I(\lambda)}, \xi_{I(\lambda),I(\lambda)}\Delta_1^{r-1}(F_T)v_J\rangle = q^{-r(T)}$ from which the second statement follows similarly. $\square$

Let $\mathcal{S} = \{(\lambda, I, J) \mid \lambda \in \Lambda^+(n, r),\ I = I_R(T),\ J = I_R(S) \text{ for } S, T \in SST(\lambda, n)\}$. The main result in [3] gives a codeterminant basis for $S_q(n, r)$.

**Theorem 5.6.** *The set $\{\xi_{A,I(\lambda)}\xi_{I(\lambda),B} \mid (\lambda, A, B) \in \mathcal{S}\}$ is a basis for $S_q(n, r)$.*

The following follows immediately from Theorems 5.5 and 5.6 and Lemma 5.4.

**Theorem 5.7.** *The map $\theta : U_q(\mathfrak{gl}_n) \to S_q(n, r)$ is surjective.*

**Remark 2.** In [1], another version of the $q$-Schur algebra is defined using structure constants arising from flags in vector spaces over a field of $q$ elements, and a surjective map from $U_q(\mathfrak{gl}_n)$ to the $q$-Schur algebra is also given in that setting.

**Corollary 5.8.** *Let $\lambda \in \Lambda^+(n, r)$. Then $V(\lambda) = \{\xi w_\lambda \mid \xi \in S_q(n, r)\}$ and the set $\{\xi_{J,I(\lambda)}w_\lambda \mid J = I_R(T),\ \text{for } T \in SST(\lambda, n)\}$ is a basis for $V(\lambda)$.*

*Proof.* The first part of the statement follows from Lemma 5.3 and Theorem 5.7 and the second part from Theorems 5.2 and 5.5 and Lemma 5.3. $\square$

We can reformulate Theorem 4.3 in terms of the $q$-Schur algebra by first defining a map $-: S_q(n,r) \to S_q(n,r)$ by

$$\overline{\eta} = \theta(\overline{u}), \text{ where } \eta = \theta(u) \in S_q(n,r),\ u \in U_q(\mathfrak{gl}_n).$$

Then a map $-: V(\lambda) \to V(\lambda)$ is given by $\overline{\xi w_\lambda} = \overline{\xi} w_\lambda$. Note that if $\xi = \theta(u)$, then $\overline{\xi w_\lambda} = \overline{\theta(u)} w_\lambda = \theta(\overline{u}) w_\lambda = \overline{u} w_\lambda = \overline{u w_\lambda}$ by Lemma 5.3.

For the next example, consider that if $u \in U_q(\mathfrak{gl}_n)$ and the expansion of $\theta(u)\xi_{I(\lambda),I(\lambda)}$ on basis elements in $S_q(n,r)$ is given by $\theta(u)\xi_{I(\lambda),I(\lambda)} = \sum_{Q \in \mathcal{R}(\lambda,n)} a_Q \xi_{Q,I(\lambda)}$, then

$$a_Q = \theta(u)\xi_{I(\lambda),I(\lambda)}(x_{Q,I(\lambda)}) = \theta(u)(x_{Q,I(\lambda)}) = \langle u v_{I(\lambda)}, v_Q \rangle. \tag{7}$$

Also note that $\overline{u^\lambda} = u^\lambda$ so that $\overline{\xi}_{I(\lambda),I(\lambda)} = \xi_{I(\lambda),I(\lambda)}$.

**Example 5.9.** Let $\lambda = (2,1)$ and let $T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$.

Then $\overline{\xi}_{(1,2,3),(1,1,2)} = \theta(\overline{F}_{T_1} u^\lambda) = \theta(\overline{F_1 F_2 u^\lambda}) = \theta(F_1 F_2 u^\lambda) = \xi_{(1,2,3),(1,1,2)}$,
and $\overline{\xi}_{(1,3,2),(1,1,2)} = \theta(F_2 F_1 u^\lambda - q F_1 F_2 u^\lambda)\xi_{I(\lambda),I(\lambda)} = \theta(F_2 F_1)\xi_{I(\lambda),I(\lambda)} - q\xi_{(1,2,3),(1,1,2)}$.

We have $\theta(F_2F_1)\xi_{I(\lambda),I(\lambda)} = \sum_{Q \in \mathcal{R}(\lambda,n)} a_Q \xi_{Q,I(\lambda)}$, where $a_Q = \langle F_2 F_1 v_{I(\lambda)}, v_Q \rangle$. Calculating $F_2F_1 v_{I(\lambda)}$ and extracting coefficients of basis elements $v_Q$, where $Q$ gives the row sequence of a row increasing tableau, yields $\theta(F_2F_1)\xi_{I(\lambda),I(\lambda)} = \xi_{(1,3,2),(1,1,2)} + q^{-1}\xi_{(1,2,3),(1,1,2)}$. Thus

$$\overline{\xi}_{(1,3,2),(1,1,2)} = \xi_{(1,3,2),(1,1,2)} - (q - q^{-1})\xi_{(1,2,3),(1,1,2)}.$$

The following theorem is a version of Theorem 4.3 in terms of elements from the $q$-Schur algebra.

**Theorem 5.10.** *Suppose that an element $\xi_T \in S_q(n,r)$ is defined for each $T \in SST(\lambda,n)$. The set $\{\xi_T w_\lambda \mid T \in SST(\lambda,n)\}$ is the global crystal basis for $V(\lambda)$ if the following properties are satisfied for each $T \in SST(\lambda,n)$:*

(1) *As a linear combination of basis elements in $\mathcal{B}_W(\lambda)$, we have $\xi_T w_\lambda = \sum_{S \in CT(\lambda,n)} \alpha_S \omega(S)$, where $\alpha_S \in \mathbb{Z}[q]$,*
(2) *$\xi_T w_\lambda \equiv \omega(T) \mod qL_W(\lambda)$,*
(3) *$\overline{\xi_T} w_\lambda = \xi_T w_\lambda$.*

*Proof.* Suppose that, for each $T \in SST(\lambda,n)$, we have $\xi_T = \theta(u_T)$ where $u_T \in U_q(\mathfrak{gl}_n)$. By Lemma 5.3, $\xi_T w_\lambda = \theta(u_T) w_\lambda = u_T w_\lambda$. We have $\overline{u_T w_\lambda} = \overline{\xi_T w_\lambda} = \xi_T w_\lambda = u_T w_\lambda$ and $u_T w_\lambda = \xi_T w_\lambda \equiv \omega(T) \mod qL_W(\lambda)$. Thus $\{u_T w_\lambda \mid T \in SST(\lambda,n)\} = \{\xi_T w_\lambda \mid T \in SST(\lambda,n)\}$ is the global crystal basis for $V(\lambda)$. $\square$

**Example 5.11.** Referring to Example 5.9, if $\lambda = (2,1)$, then the set

$$\{\xi_{(1,2,3),(1,1,2)}w_\lambda, (\xi_{(1,3,2),(1,1,2)} + q^{-1}\xi_{(1,2,3),(1,1,2)})w_\lambda\}$$

is the portion of the global crystal basis corresponding to the weight space $V(\lambda)^\chi$, where $\chi = (1,1,1)$.

## 6. Relationsips between bases

We shall say that a tableau $T$ is diagonally related to a $\lambda$-tableau $S$, $T \rhd_d S$, if $S$ can be obtained from $T$ by exchanging an entry $a$ in $T$ with an entry $b > a$ where $a$ sits in a row below $b$ and in a column left of $b$. Define $\rhd_D$ to be the partial order defined by extending $\rhd_d$ reflexively and transitively.

**Example 6.1.** We have $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \rhd_d \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \rhd_d \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \rhd_D \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$.

Recall that if $T$ has row sequence $Q \in I(n,r)$, we denote the column sequence of $T$ by $Q^t$.

**Lemma 6.2.** *Let $\lambda \in \Lambda^+(n,r)$ and suppose that $x_{M,I_C(\lambda)} = \sum_{K \in \mathcal{R}(\lambda,n)} a_k x_{K,I(\lambda)}$ as a linear combination of basis elements. Then, if $a_K \neq 0$, we have $T_{K^t} \sim_R W \rhd_D T_M$ for some tableau $W$.*

*Proof.* We will use a specific recipe for rewriting $x_{M,I_C(\lambda)}$ as a linear combination of basis elements. Starting with $i = 1$, and the left-most $x_{mi}$ in $x_{M,I_C(\lambda)}$, use the relations (3) to move $x_{mi}$ left of all $x_{sj}$ where $x_{sj}$ sits left of $x_{mi}$ and $j > i$. Repeat this procedure for $i = 2, \ldots, \mu_1$, where $\mu = (\mu_1, \ldots, \mu_{\lambda_1})$ is the conjugate partition, and then for each of the resulting summands to get

$$x_{M,I_C(\lambda)} = \sum_B c_B x_{B,I(\lambda)}. \tag{8}$$

Now rewrite each $x_{B,I(\lambda)}$ in the sum using the second of the relations (3) to get $\sum_K a_K x_{K,I(\lambda)}$ where each $(K, I(\lambda))$ in the sum satisfies $(K, I(\lambda))_0 = (K, I(\lambda))$.

If $a_K \neq 0$, then one possibility is that $(M, I_C(\lambda))_0 = (K, I(\lambda))$, in which case $T_{K^t} \sim_R T_M$, and in this case, $W = T_{K^t}$. Otherwise, the fourth property of relations (3) was used at least once in the above procedure which resulted in $x_{K,I(\lambda)}$ in the sum. There is then an $x_{B,I(\lambda)}$ in the first sum (8) with $(B, I(\lambda))_0 = (K, I(\lambda))$ (in other words, $T_{K^t} \sim_R T_{B^t}$) and the fourth relation was used at least once in rewriting $x_{M,I_C(\lambda)}$ to get $x_{B,I(\lambda)}$ in the sum (8). We will show that $T_{B^t} \rhd_D T_M$. Since the fourth relation was applied to $x_{M,I_C(\lambda)}$, we have

$$\begin{aligned} x_{M,I_C(\lambda)} &= \cdots x_{m_1,j_1} \cdots x_{m_2,j_2} \cdots \\ &= \alpha(q) x_{m_2,j_1} \cdots x_{m_1,j_2} \cdots + \text{ other terms} \\ &= \alpha(q) x_{M_1,I_C(\lambda)} + \text{ other terms,} \end{aligned}$$

where $m_1 > m_2$ and $j_1 > j_2$. Since $j_1 > j_2$, in the tableau $T_M$ we have $m_1 > m_2$ and $m_1$ sits southwest of $m_2$. It follows that $T_{M_1} \rhd_d T_M$. Either $(B, I_C(\lambda))_0 = (M, I_C(\lambda))_0$ or the fourth relation can be applied again to $x_{M_1,I_C(\lambda)}$ to get $T_{M_2} \rhd_d$

$T_{M_1} \rhd_d T_M$. Inductively, we have $T_{B^t} \rhd_d \cdots \rhd_d T_{M_1} \rhd_d T_M$, so that $T_{B^t} \rhd_D T_M$. Consequently, $T_{K^t} \sim_R T_{B^t} \rhd_D T_M$. □

**Lemma 6.3.** *Suppose that $S \in RT(\lambda, n)$ and let $Q$ denote the row sequence of $S$. Then, as a $\mathbb{Q}[q, q^{-1}]$-linear combination of basis elements in $\mathcal{B}_W(\lambda)$, we have $\xi_{Q,I(\lambda)} w_\lambda = \sum_{T \in CT(\lambda,n)} b_T \omega(T)$, where if $b_T \neq 0$, then $S \sim_R W_1 \rhd_D W_2 \sim_C T$, for some $\lambda$-tableaux $W_1$ and $W_2$.*

*Proof.* We have $\xi_{Q,I(\lambda)} w_\lambda = \sum_{A \in I(n,r)} \xi_{Q,I(\lambda)}(x_{A,I_C(\lambda)})\omega(T_A)$. By Lemma 6.2, for each $A$ in the sum we have $x_{A,I_C(\lambda)} = \sum_{(K,I(\lambda)) \in \mathcal{J}} c^A_K x_{K,I(\lambda)}$, where $c^A_K = 0$ unless $T_{K^t} \sim_R W_1 \rhd_D T_A$. Since $(K, I(\lambda)), (Q, I(\lambda)) \in \mathcal{J}$, $\xi_{(Q,I(\lambda))}(x_{(K,I(\lambda))}) = 0$ unless $K = Q$. Thus $\xi_{Q,I(\lambda)} w_\lambda = \sum_{A \in I(n,r)} c^A_Q \omega(T_A)$, where for each $A$ in the sum, $S \sim_R W_1 \rhd_D T_A$. It may be that $T_A$ is not column increasing, in which case $\omega(T_A) = \pm\omega(T)$, where $T_A \sim_C T$ and $T$ is column increasing, so that $\omega(T) \in \mathcal{B}_W(\lambda)$. □

**Lemma 6.4.** *Suppose that $T \in SST(\lambda, n)$ and that $T \sim_R W \rhd_D S$ for $\lambda$-tableaux $W$ and $S$. Then $T \leq_C S$ and, if $U$ is equal to the $\lambda$-tableau obtained by rewriting the columns of $S$ in increasing order, then $T \leq_C U$.*

*Proof.* If $T \sim_R W$, then $T \leq_C W$. Furthermore, if $W \rhd_d W_1 \rhd_d \cdots \rhd_d W_k \rhd_d S$, where $W \neq S$, an inductive argument shows that $W \leq_C S$ so that $T \leq_C S$. To see that $T \leq_C U$, where $U$ comes from $S$ by rewriting its columns to be increasing, consider the left-most column where $T$ and $U$ differ. Since all columns prior to this column contain the same entries in both $T$ and $U$, the smallest entry in this column of $U$ that is different from one in $T$ must have arisen through a row exchange with an entry larger than one in $T$, possibly combined with a number of diagonal exchanges, which again increase entries. Thus the column sequence of $T$ associated to this column is less than that of $U$ and so $T \leq_C U$. □

**Corollary 6.5.** *Let $T \in RT(\lambda, n)$ and let $Q$ be the row sequence of $T$. Then $\xi_{Q,I(\lambda)} w_\lambda = \sum_{T_A \in CT(\lambda,n)} b_A \omega(T_A)$, where for each $T_A$ in the sum, $T_{Q^t} \leq_C T_A$ and $b_{Q^t} = q^{\epsilon(Q^t, I_C(\lambda))}$.*

*Proof.* We have $\xi_{Q,I(\lambda)} w_\lambda = \sum_{A \in I(n,r)} \xi_{Q,I(\lambda)}(x_{A,I_C(\lambda)})\omega(T_A)$ and $\xi_{Q,I(\lambda)}(x_{A,I_C(\lambda)})$ contributes to the coefficient $b_{Q^t}$ of $\omega(T_Q)$ if and only if $T_A = T_{Q^t}\sigma$ for some $\sigma \in C(\lambda)$. However, using Lemma 6.2, $\xi_{Q,I(\lambda)}(x_{Q^t\sigma,I_C(\lambda)}) = 0$ for $\sigma \in C(\lambda)$ unless $\sigma$ is the identity permutation. Thus

$$b_{Q^t} = \xi_{Q,I(\lambda)}(x_{Q^t,I_C(\lambda)}) = \xi_{Q,I(\lambda)}(q^{\epsilon(Q^t,I_C(\lambda))} x_{Q,I(\lambda)} + \sum_{(S,T)} x_{S,T}),$$

where the pairs $(S, T)$ in the sum satisfy $(S, T)_0 = (S, T)$ and $(S, T) > (Q^t, I_C(\lambda))$ by Lemma 3.1. It follows that $b_{Q^t} = q^{\epsilon(Q^t,I_C(\lambda))} \neq 0$. □

Let $\mathcal{SST}(\lambda, n) = \{Q \in I(n,r) \mid Q = I_R(T) \text{ for some } T \in SST(\lambda, n)\}$. An immediate consequence of the following theorem is that the global crystal basis and Carter-Lusztig basis for $V(\lambda)$ are related by an upper triangular invertible matrix.

**Theorem 6.6.** *Let $T$ be a semistandard $\lambda$-tableau with row sequence $J$ and suppose that $a(T)w_\lambda = \sum_{Q \in \mathcal{SST}(\lambda,n)} a_Q \xi_{Q,I(\lambda)} w_\lambda$ is the expansion of $a(T)w_\lambda$ in the basis $\{\xi_{Q,I(\lambda)} w_\lambda \mid Q \in SST(\lambda, n)\}$. Then*

(1) *$a_J = q^{-s(T)}$, and*
(2) *if $a_Q \neq 0$, then $\omega(T_{Q^t})$ and $\omega(T)$ have the same weight and $T \leq_C T_{Q^t}$.*

*Proof.* The fact that each $Q$ with $a_Q \neq 0$ corresponds to $\omega(T_{Q^t})$ with the same weight as $\omega(T)$ follows from Lemma 4.5 combined with Lemma 6.3. Suppose that some $Q$ in the sum has $T_{Q^t} < T$ and choose $K$ so that $K^t$ is minimal with this property. By Corollary 6.5, when each $\xi_{Q,I(\lambda)} w_\lambda$ is written as a $\mathbb{Q}[q, q^{-1}]$-linear combination of basis elements in $\mathcal{B}_W(\lambda)$, $\omega(T_{K^t})$ only appears in $\xi_{K,I(\lambda)} w_\lambda$, and it appears with non-zero coefficient and so appears with non-zero coefficient in the sum $a(T)w_\lambda$, which is not possible by Lemma 4.5.

Thus $a(T)w_\lambda = \sum_{Q \in \mathcal{SST}(\lambda,n)} a_Q \xi_{Q,I(\lambda)} w_\lambda = a_J \xi_{J,I(\lambda)} w_\lambda + \sum_{Q \in \mathcal{SST}(\lambda,n)} a_Q \omega(T_{Q^t})$, where each $Q$ in the sum has $T_{Q^t} > T$. But $a_J \xi_{J,I(\lambda)} w_\lambda = q^{\epsilon(J^t, I_C(\lambda))} a_J \omega(T) + \sum_B a_B \omega(T_B)$, where each $\omega(T_B) \in \mathcal{B}_W(\lambda)$ with $T_B > T$. Furthermore, $a(T)w_\lambda = \omega(T) + \sum_L c_L \omega(T_L)$ where $T_L > T$. It follows that $a_J = q^{-\epsilon(J^t, I_C(\lambda))}$.

Write $J^t = (j_1, j_2, \ldots, j_r)$ and $I_C(\lambda) = (i_1, i_2, \ldots, i_r)$. If $i_a = i_b$, then $j_a$ and $j_b$ belong to the same row and, since $T$ is semistandard, $j_a < j_b$. It follows that

$$\epsilon(J^t, I_C(\lambda)) = \{(a,b) \mid a < b,\ j_a = j_b,\ i_a > i_b\}.$$

If $j_a$ belongs to column $k$ of $T$ and $j_b$ belongs to column $\ell$, then $j_a = T_{i_a k}$ and $j_b = T_{i_b \ell}$ and, since $T$ is semistandard, $\ell < k$ whenever $j_a = j_b$ and $a < b$. Thus $\epsilon(J^t, I_C(\lambda)) = \{(k, \ell, i_a, i_b) \mid \ell < k,\ T_{i_a k} = T_{i_b \ell},\ i_a > i_b\} = s(T)$. □

## 7. An algorithm for writing the global crystal basis in terms of elements from the $q$-Schur algebra

The algorithm from [13] allows us to write each element of the global crystal basis vectors from $V(\lambda)$ in terms of elements $a(T)w_\lambda$ from the Leclerc-Toffin basis. The map $\theta : U_q(\mathfrak{gl}_n) \to S_q(n,r)$ can then be exploited to write each $a(T)w_\lambda$ in terms of elements from the $q$-Schur algebra. We first establish two lemmas which shorten computation time.

If $a(T) = F_{i_1}^{(r_1)} \cdots F_{i_s}^{(r_s)} \in U_q(\mathfrak{gl}_n)^-$, define $b(T) = \tau(a(T)) = E_{i_s}^{(r_s)} \cdots E_{i_1}^{(r_1)} \in U_q(\mathfrak{gl}_n)^+$. Since it is often easier to find $\langle v_{I(\lambda)}, b(T)v_Q\rangle$ than $\langle a(T)v_{I(\lambda)}, v_Q\rangle$, the following lemma is quite useful.

**Lemma 7.1.** *Let $T \in SST(\lambda, n)$, and let $Q$ denote the row sequence of $T$. Then*

$$\langle a(T)v_{I(\lambda)}, v_Q\rangle = q^{r(T)-s(T)} \langle v_{I(\lambda)}, b(T)v_Q\rangle.$$

*Proof.* Using Lemma 5.1 we have $\langle a(T)v_{I(\lambda)}, v_Q\rangle = \langle v_{I(\lambda)}, \Delta_1^{r-1}(b(T))v_Q\rangle$. By (1),

$$\begin{aligned}\langle v_{I(\lambda)}, \Delta_1^{r-1}(b(T))v_Q\rangle &= q^{\beta(Q)-\beta(I(\lambda))}\langle \Delta_1^{r-1}(a(T))v_{I(\lambda)}, v_Q\rangle \\ &= q^{\beta(Q)-\beta(I(\lambda))}\langle v_{I(\lambda)}, b(T)v_Q\rangle.\end{aligned}$$

Now, $\beta(Q)$ counts the number of pairs $(a, b)$ in $T$ where $a$ and $b$ belong to the same row but $a < b$ plus the pairs where $a \neq b$ and $b$ belongs to a row below $a$. Furthermore, $\beta(I(\lambda)$ counts the pairs $(a, b)$ in $T$ where $b$ sits in a row below $a$. Thus, $\beta(Q) - \beta(I(\lambda)) = r(T) - s(T)$. □

The following lemma allows us to classify the $Q \in \mathcal{R}(\lambda, n)$ that yield a non-zero coefficient $a_Q$ in the linear combination $\theta(a(T))\xi_{I(\lambda),I(\lambda)} = \displaystyle\sum_{Q\in\mathcal{R}(\lambda,n)} a_Q\xi_{Q,I(\lambda)}$. We first give a simple example to illustrate the result.

**Example 7.2.** Let $\lambda = (2, 1)$ and consider the $\lambda$-tableau $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. Then

$$a(T)v_1 \otimes v_1 \otimes v_2 = v_1 \otimes v_3 \otimes v_2 + q^{-1}v_1 \otimes v_2 \otimes v_3 + qv_3 \otimes v_1 \otimes v_2 + v_2 \otimes v_1 \otimes v_3.$$

Consider the tableaux $T_{M^t}$ arising from the 3-tuples $M$ that appear in the linear combination $a(T)v_{I(\lambda)} = \sum_M a_M v_M$. We have

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \sim_R \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \text{ and } \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \rhd_D \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \sim_R \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

**Lemma 7.3.** *Suppose that $T$ is a semistandard $\lambda$-tableau. If $\langle a(T)v_{I(\lambda)}, v_K\rangle \neq 0$, then $T \rhd_D W \sim_R T_{K^t}$ for some $\lambda$-tableau $W$.*

*Proof.* We will show that $a(T)v_{I(\lambda)} = \sum_K a_K v_K$ where, for each $a_K \neq 0$, we have $T \rhd_D W \sim_R T_{K^t}$ for some $\lambda$-tableau $W$. To make the connection with Young tableau more readily apparent, we will associate $v_M \in V^{\otimes r}$ with the tableau $T_{M^t}$ (not to be confused with $\omega(T_{M^t}) \in W(\lambda)$ which would be zero if $T_{M^t}$ contained two equal column entries, while the corresponding $v_M$ would not be zero). Instead of writing $a(T)v_{I(\lambda)}$, for instance, we will write $a(T)T(\lambda)$ and keep track of the effect of applying the $F_i$'s in this way. We write $F_iT_{M^t} = \sum_B a_BT_{B^t}$ when $F_iv_M = \sum_B a_Bv_B$.

The proof is by induction on the number of entries $\ell$ in $T$, $1 \leq \ell \leq n$, that belong to a row $r$ with $\ell \neq r$. Suppose first that there is one such $\ell$ and let $r$ be the highest row in $T$ in which there is an $\ell$ with $r < \ell$. Then all $\ell$'s in $T$ appear below row $r-1$ and above row $\ell+1$ and $a(T)T(\lambda) = F^{(k_0)}_{\ell-1,\ell}F^{(k_1)}_{\ell-2,\ell-1}\cdots F^{(k_j)}_{r,r+1}T(\lambda)$. Suppose that $T_0$ is the tableau that comes from $T$ by changing all $\ell$'s above row $\ell$ to $\ell-1$, $T_1$ is the tableau that comes from $T_0$ by changing all $\ell-1$'s above row $\ell-1$ in $T_0$ to $\ell-2$, …, $T_j$ comes from changing all $r+2$'s above row $r+2$ of $T_{j-1}$ to $r+1$'s (in other words, $T_j$ is the same as $T(\lambda)$ except that the $k_j$ rightmost $r$'s in row $r$ have been changed to $r+1$'s).

Then $F^{(k_j)}_{r,r+1}T(\lambda) = T_j + \sum_\alpha T_\alpha$ where the sum $\sum_\alpha T_\alpha$ runs over the non-semistandard $T_\alpha$ that come from $T(\lambda)$ by replacing $k_j$ entries in row $r$ with $r+1$; in particular, $T_\alpha \sim_R T_j$ for each $\alpha$. Below, we will use the fact that, if $F_iS = \sum_k a_kT_k$, for a tableau $S$, and $S \sim_R W$, then whenever $a_m \neq 0$ in the sum $F_i W = \sum_m a_m T_m$, we have $T_m \sim_R T_k$ for some $a_k \neq 0$.

Applying $F^{(k_{j-1})}_{r+1,r+2}$ to $T_j$ yields a sum of tableaux that come from replacing $k_{j-1}$ entries equal to $r+1$ in $T_j$ with $r+2$. If we change the $k_j$ rightmost $r+1$'s in row $r$ of $T_j$ to $r+2$'s and the rightmost $k_{j-1} - k_j$ entries equal to $r+1$ in row $r+1$ of $T_j$, to $r+2$'s, we obtain $T_{j-1}$. The other tableaux in the sum $F^{(k_j-1)}_{r+1,r+2} T_j$ are either row equivalent to $T_j$ or come from changing $t < k_j$ entries equal to $r+1$ in row $r$ to $r+2$ and $k_{j-1} - t$ entries in row $r+1$ to $r+2$. If such a tableau $T_\beta$ is weakly row increasing, then $T_{j-1} \rhd_D T_\beta$ by interchanging a series of $r+1$'s in row $r+1$ with $r+2$'s in row $r$. If not, $T_\beta$ will be row equivalent to such a tableau.

None of the tableaux obtained from applying $F^{(k_{j-1})}_{r+1,r+2}$ to $\sum_\alpha T_\alpha$ will be row increasing, but since $T_j \sim_R T_\alpha$, any $S$ in the sum $F^{(k_{j-1})}_{r+1,r+2} T_\alpha$ will be row equivalent to some $T_\beta$ in the sum $F^{(k_{j-1})}_{r+1,r+2} T_j$. Thus $T_{j-1} \rhd_D W \sim_R T_\beta \sim_R S$ for some $\lambda$-tableau $W$. Write

$$F^{(k_0)}_{\ell-1,\ell} F^{(k_1)}_{\ell-2,\ell-1} \cdots F^{(k_j)}_{r,r+1} T(\lambda) = F^{(k_0)}_{\ell-1,\ell}(T_0 + \sum_\kappa a_\kappa T_\kappa), \tag{9}$$

where each weakly row increasing $T_\kappa$ in the sum $\sum_\kappa a_\kappa T_\kappa)$ has $T_0 \rhd_D T_\kappa$ and all other tableaux in the sum are row equivalent to a tableau of that sort. Now, $F^{(k_0)}_{\ell-1,\ell} T_0 = T + \sum_\alpha c_\alpha T_\alpha$ where every weakly row increasing $T_\alpha$ has $T \rhd_D T_\alpha$ and the other tableaux are row equivalent to one of these, using the same argument as above.

For the remaining tableaux $T_\kappa$ in the sum (9), consider first those $T_\kappa$ that are weakly row increasing. Since every entry in $T_0$ that sits above row $\ell - 1$ is less than or equal to $\ell - 1$ and $T_0 \rhd_D T_\kappa$, every entry in row $\ell - 1$ of $T_\kappa$ is equal to $\ell - 1$. Thus, there are exactly $k_0$ entries equal to $\ell - 1$ above row $\ell - 1$. One tableau that arises from applying $F^{(k_0)}_{\ell-1,\ell}$ to $T_\kappa$ is the tableau $S$ where $S$ comes from changing the $k_0$ entries equal to $\ell - 1$ above row $\ell - 1$ in $T_\kappa$ to $\ell$'s; certainly $S$ is weakly row increasing.

Then, if $T_0 \rhd_d V_1 \rhd_d \cdots \rhd_d V_r \rhd_d T_\kappa$, we have $T \rhd_d V_1' \rhd_d \cdots V_r' \rhd_d S$ where $V_1'$ comes from $T$ by swapping the entries in the same boxes that were swapped to get from $T_1$ to $V_1$ (or not at all if this would mean swapping two entries equal to $\ell$), etc. One may also obtain weakly row increasing tableaux from $T_\kappa$ by using $F^{(k_0)}_{\ell-1,\ell}$ to change some $\ell - 1$'s in row $\ell - 1$ to $\ell$'s and some above row $\ell - 1$ to $\ell$'s. Suppose that $V$ is such a tableau. We have $S \rhd_D V$, by exchanging a series of $\ell - 1$'s with $\ell$'s, so $T \rhd_D S \rhd_D V$. If $F^{(k_0)}_{\ell-1,\ell}$ is applied to a non-weakly row increasing $T_{\kappa'}$ from the $\sum_\kappa \alpha_\kappa T_\kappa$ portion of the sum (9), then, since there is a weakly row increasing $T_\kappa$ in the sum with $T_\kappa \sim_R T'_\kappa$, any tableau $V$ in the sum $F^{(k_0)}_{\ell-1,\ell} T'_\kappa$ will also satisfy $T \rhd_D W \sim_R V$ for some tableau $W$. Thus, all non-weakly row increasing tableaux in the sum $F^{(k_0)}_{\ell-1,\ell}(T_0 + \sum_\kappa a_\kappa T_\kappa)$ are either row equivalent to $T$ or row equivalent to a tableau $W$ with $T \rhd_D W$. This shows that, for $T$ with weight $\chi$ with $\chi_\ell > \lambda_\ell$ and $\chi_i = \lambda_i$ for $i \neq \ell$ we have $a(T)T(\lambda) = T + \sum_\alpha c_\alpha T_\alpha$ where for each $c_\alpha \neq 0$ we have $T \rhd_D W \sim_R T_\alpha$ for some tableau $W$.

For an arbitrary $T$, suppose that $\ell$ is the smallest entry in $T$ that is in a row $x$ with $x < \ell$ and suppose that $r$ is the smallest row that contains an $\ell$ with $r < \ell$. Let $T_0$ be the tableau that comes from $T$ by replacing all $\ell$'s above row $\ell$ with the row number to which they belong. Then $a(T)T(\lambda) = F^{(k_0)}_{\ell-1,\ell} \cdots F^{(k_j)}_{r,r+1} a(T_0)T(\lambda)$ so by induction we have $a(T)T(\lambda) = F^{(k_0)}_{\ell-1,\ell} \cdots F^{(k_j)}_{r,r+1}(T_0 + \sum_\beta a_\beta T_\beta)$, where each $T_\beta$ has $T_0 \rhd_D W \sim_R T_\beta$.

There are no entries equal to $x$ above row $x$ for any $x$ with $r \leq x \leq \ell$ in $T$, so the same is true of $T_0$. Thus, an argument similar to that above shows that $F^{(k_0)}_{\ell-1,\ell} \cdots F^{(k_j)}_{r,r+1} T_1 = T_J + \sum_\alpha c_\alpha T_\alpha$ where $T \rhd_D W \sim_R T_\alpha$ for some tableau $W$, for each $c_\alpha \neq 0$ in the sum.

We can also use a similar argument to that above to prove that, for each weakly row increasing $T_\beta$ in the sum above, $F^{(k_0)}_{\ell-1,\ell} \cdots F^{(k_j)}_{r,r+1} T_\beta = \sum_i a_i T_i$, where each $T_i$ has $T \rhd_D W \sim_R T_i$. Since the other tableaux in the sum are row equivalent to weakly row increasing tableaux, we obtain the result for all $T_\beta$ in the sum. □

From the above two lemmas and (7), we obtain the following corollary.

**Corollary 7.4.** *Suppose that $T$ is a semistandard $\lambda$-tableau. Then*

$$\theta(a(T))\xi_{I(\lambda),I(\lambda)} = \sum_{Q \in \mathcal{R}(\lambda,n)} a_Q \xi_{Q,I(\lambda)},$$

*where $a_Q = q^{r(T)-s(T)} \langle v_{I(\lambda)}, b(T)v_Q \rangle$ and, if $a_Q \neq 0$, then $T \rhd_D T_{Q^t}$.*

In light of the above, we can obtain a linear combination

$$a(T)w_\lambda = \theta(a(T))\xi_{I(\lambda),I(\lambda)} w_\lambda = \sum_{Q \in \mathcal{R}(\lambda,n)} a_Q \xi_{Q,I(\lambda)} w_\lambda$$

by considering all $T_{Q^t} \in RT(\lambda, n)$ with $T \rhd_D T_{Q^t}$. If for any $Q$ in this sum $T_{Q^t}$ is not semistandard, rewrite $\xi_{Q,I(\lambda)} w_\lambda$ as a linear combination of basis elements $\xi_{Q,I(\lambda)} w_\lambda = \sum_{(J,I(\lambda)) \in \mathcal{J}} a_J \xi_{J,I(\lambda)} w_\lambda$, where each $J$ from the sum gives a semistandard $\lambda$-tableau $T_{J^t}$. We can then use the algorithm from [13] to write each global crystal basis vector as a linear combination of vectors from the Leclerc-Toffin basis, which in turn gives a linear combination of elements from the $q$-Schur algebra version of the Carter-Lusztig basis.

**Example 7.5.**

1. Let $\lambda = (2, 1)$. The weakly row increasing $\lambda$-tableaux with weight $\chi = (1, 1, 1)$ are $T_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$, $T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, $T_3 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$.

In this case, $a(T_1)w_\lambda \equiv \omega(T_1) \mod qL_W(\lambda)$ and $a(T_2)w_\lambda \equiv \omega(T_2) \mod qL_W(\lambda)$ so these are the global basis vectors for the weight space $V(\lambda)^\chi$, where $\chi = (1, 1, 1)$.

We have $a(T_2)w_\lambda = \xi_{(1,2,3),(1,1,2)}w_\lambda$ and $a(T_1)w_\lambda = (\xi_{(1,3,2),(1,1,2)} + a_1\xi_{(1,2,3),(1,1,2)})w_\lambda$. Computing $B(T_1)v_1 \otimes v_1 \otimes v_2$ and employing Corollary 7.4 yields $a_1 = q^{-1}$. Compare with Example 5.11.

2. Let $\lambda = (3,1)$ and consider the weakly row increasing $\lambda$-tableaux of weight $\chi = (1,2,1,0)$: $T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$, $T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, $T_3 = \begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$. Let $G(T_1)$ and $G(T_2)$ denote the global crystal basis vectors for the weight space $V(\lambda)^\chi$.

We have $a(T_2)w_\lambda = \xi_{(1,2,2,3),(1,1,1,2)}w_\lambda = G(T_2)$ and, since $T_1 \rhd_D T_2$, $a(T_1)w_\lambda = q^{-1}\xi_{(1,2,3,2),(1,1,1,2)} + a\xi_{(1,2,2,3),(1,1,1,2)}$, $a \in \mathbb{Q}[q,q^{-1}]$. But

$$q^{r(T_2)-s(T_2)}\langle v_{I(\lambda)}, B(T_1)v_1 \otimes v_2 \otimes v_2 \otimes v_3\rangle = q^2(q^{-2} + q^{-4}),$$

so

$$a(T_1)w_\lambda = q^{-1}\xi_{(1,2,3,2),(1,1,1,2)}w_\lambda + (1+q^{-2})\xi_{(1,2,2,3),(1,1,1,2)}w_\lambda.$$

Since $a(T_1)w_\lambda \equiv (\omega(T_1) + (1+q^2)\omega(T_2)) \mod qL_W(\lambda)$, $a(T_1)w_\lambda$ is not a global crystal basis vector. However, since $a(T_1)w_\lambda - a(T_2)w_\lambda \equiv \omega(T_1) \mod qL_W(\lambda)$,

$$G(T_1) = a(T_1)w_\lambda - a(T_2)w_\lambda = q^{-2}\xi_{(1,2,2,3),(1,1,1,2)}w_\lambda + q^{-1}\xi_{(1,2,3,2),(1,1,1,2)}w_\lambda.$$

It follows that

$$\{\xi_{(1,2,2,3),(1,1,1,2)}w_\lambda,\ q^{-2}\xi_{(1,2,2,3),(1,1,1,2)}w_\lambda + q^{-1}\xi_{(1,2,3,2),(1,1,1,2)}w_\lambda\}$$

is the portion of the global crystal basis for the weight space $V(\lambda)^\chi$.

## References


[1] A.A. Beilinson, G. Lusztig, and R. MacPherson, *A geometric setting for the quantum deformation of $GL_n$*, Duke Math. J. **61** (1990), 655–677.

[2] J. Brundan, *Dual canonical bases and Kazhdan-Lusztig polynomials*, J. Algebra **306** (2006), 17–46.

[3] G. Cliff and A. Stokke, *Codeterminants and q-Schur algebras*, Algebr. Represent. Theory **13** (2010), no. 1, 43–60.

[4] W.A. de Graaf, *An algorithm to compute the canonical basis of an irreducible module over a quantized enveloping algebra*, LMS J. Comput. Math. **6** (2003), 105–118.

[5] R. Dipper and S. Donkin, *Quantum $GL_n$*, Proc. London Math. Soc. **63** (1991), 165–211.

[6] R. Dipper and G. James, *The q-Schur algebras*, Proc. London Math. Soc. **59** (1989), 23–50.

[7] J.A. Green, *Polynomial Representations of $GL_n$*, Lecture Notes in Math., vol. 830, Berlin/Heidelberg, New York, 1980.

[8] J. Hong and S-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Graduate Studies in Mathematics, vol. 42, American Mathematical Society, Providence, Rhode Island, 2002.

[9] R.Q. Huang and J.J. Zhang, *Standard basis theorem for quantum linear groups*, Adv. Math. **102** (1993), 202–229.

[10] J.C. Jantzen, *Lectures on Quantum Groups*, Graduate Studies in Mathematics, vol. 6, American Mathematical Society, Providence, Rhode Island, 1996.

[11] D. Janzen, S. Phillips, and A. Stokke, *A combinatorial description of the quantum Désarménien matrix via the q-Schur algebra*, J. Algebra **304** (2006), 906–926.

[12] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.

[13] B. Leclerc and P. Toffin, *A simple algorithm for computing the global crystal basis of an irreducible $U_q(sl_n)$-module*, Internat. J. Algebra Comput. **10** (2000), no. 2, 191–208.

[14] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Am. Math. Soc. **3** (1990), no. 2, 447 – 498.
[15] Yu. I. Manin, *Quantum Groups and Noncommutative Geometry*, Université de Montréal, Montreal, 1988.
[16] S. Martin, *Schur Algebras and Representation Theory*, Cambridge Univ. Press, Cambridge, 1993.
[17] B. Parshall and J.-p. Wang, *Quantum Linear Groups*, Memoirs Amer. Math. Soc., vol. 89, Amer. Math. Soc., Providence, 1991.
[18] A. Stokke, *A quantum version of the Désarménien matrix*, J. Alg. Combinatorics **22** (2005), 303–316.
[19] E. Taft and T. Towber, *Quantum deformation of flag schemes and Grassman schems I. A $q$-deformation of the shape algebra for $GL(n)$*, J. Algebra **142** (1991), 1–36.

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