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Note

Jeu-de-taquin promotion and a cyclic sieving phenomenon for semistandard hook tableaux

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ABSTRACT

Jeu-de-taquin promotion yields a bijection on the set of semistandard λ-tableaux with entries bounded by k. In this note, we determine the order of jeu-de-taquin promotion on the set of semistandard hook tableaux $CST((n-m, 1^m), k)$, with entries bounded by k, and on the set of semistandard hook tableaux with fixed content α, $CST((n-m, 1^m), k, \alpha)$. We give a bijection between $CST((n-m, 1^m), k, \alpha)$ and a suitable set of standard hook tableaux that behaves nicely with respect to jeu-de-taquin promotion and use the bijection to give a cyclic sieving phenomenon for $CST((n-m, 1^m), k, \alpha)$.

1. Introduction

Schützenberger's jeu-de-taquin promotion operator is a bijection on the set of standard λ-tableaux $SYT(\lambda)$ [11,12]. Haiman [5] showed that n iterations of jeu-de-taquin promotion fixes every standard rectangular tableau in $SYT(\lambda)$, where $\lambda = (c^r)$ is a partition of n. Rhoades [9] studied jeu-de-taquin promotion on column-strict, or semistandard, λ-tableaux $CST(\lambda, k)$, with entries bounded by k, and on semistandard λ-tableaux $CST(\lambda, k, \alpha)$ with a particular content α, and showed that the order of promotion on $CST((c^r), k)$ is equal to k unless the Young diagram of shape λ is a single row or column and $k = cr$. For general shapes, the order of promotion on $SYT(\lambda)$ and $CST(\lambda, k)$ is not known.

Rhoades also revealed a connection between jeu-de-taquin promotion and the cyclic sieving phenomenon (CSP) of Reiner, Stanton, and White [8]. Let X be a finite set, $C = \langle g \rangle$ a finite cyclic group of order N that acts on X, and $X(q)$ a polynomial with integer coefficients. The triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon (CSP) if, for any integer d,

$$X(\omega^d) = |\{x \in X \mid g^d \cdot x = x\}|,$$

where $\omega = e^{2\pi i/N}$ is a primitive Nth root of unity. Cyclic sieving phenomena have been exhibited in many different settings (see, for example, [1,4,6,7]).

Using Kazhdan–Lusztig theory, Rhoades [9] proved that $(SYT(c^r), C, X(q))$ exhibits the CSP, where $C = \langle j \rangle$ acts on $SYT(c^r)$ by jeu-de-taquin promotion and $X(q)$ is the q-analogue of the Frame–Robinson–Thrall hook-length formula [3]. Rhoades also gave a CSP for semistandard rectangular tableaux which involves the cyclic action of the jeu-de-taquin operator and the q-analogue of the hook-content formula. For a survey of the literature on the cyclic sieving phenomenon, see [10].

In Section 3, we define a bijection between the set of semistandard hook tableaux $CST((n-m, 1^m), k, \alpha)$, with entries bounded by k and fixed content α, and a set of standard λ'-tableaux, $SYT(\lambda')$, where the partition λ' is determined from

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$\lambda = (n-m, 1^m)$ and α. Our bijection behaves nicely with respect to jeu-de-taquin promotion, and we use it to prove that the order of promotion on $CST((n-m, 1^m), k, \alpha)$ is $p(\alpha) \cdot (n(\alpha) - 1)$, where $p(\alpha)$ is the cyclic symmetry of α and $n(\alpha)$ is the number of nonzero entries in α. We also prove that the order of promotion on $CST((n-m, 1^m), k)$ is equal to $k \cdot \mathrm{lcm}(m+1, m+2, \ldots, k-2, k-1)$ when $m+1 < k \le n$ and $k \cdot \mathrm{lcm}(m+1, m+2, \ldots, n-2, n-1)$ when $k \ge n$.

In Section 4, we use our bijection combined with Reiner, Stanton, and White's CSP result for k-element subsets of a set $\{1, 2, \ldots, n\}$ to prove that $(X, C, X(q))$ exhibits the CSP, where C acts on $X = CST((n-m, 1^m), k, \alpha)$ by the $p(\alpha)$th power of promotion and $X(q)$ is the q-analogue of $\binom{n(\alpha)-1}{m}$.

2. Young tableaux and jeu-de-taquin promotion

An r-tuple of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a *partition* of n if $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Throughout, λ shall be a fixed partition of a positive integer n.

The *Young diagram of shape* $\lambda = (\lambda_1, \ldots, \lambda_r)$ consists of r left-justified rows where the ith row contains λ_i boxes. A λ*-tableau* T is obtained by filling the boxes of the Young diagram of shape λ with positive integers. The entry in the ith row and jth column of T will be denoted by $T(i, j)$.

A λ-tableau is *semistandard*, or column strict, if the entries in its columns are strictly increasing from top to bottom and the entries in its rows are weakly increasing from left to right. We denote the set of semistandard λ-tableaux with entries less than or equal to a positive integer k by $CST(\lambda, k)$.

A k-tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a *composition* of n of length k if $\alpha_i \ge 0$ for $1 \le i \le k$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$. A λ-tableau T has content $\alpha = (\alpha_1, \ldots, \alpha_k)$ if T has α_i entries equal to i for $1 \le i \le k$.

A λ-tableau is *standard* if its columns and its rows are strictly increasing and its content is equal to $\alpha = (1^n)$. We denote the set of standard λ-tableaux by $SYT(\lambda)$ and the set of semistandard λ-tableaux with content $\alpha = (\alpha_1, \ldots, \alpha_k)$ by $CST(\lambda, k, \alpha)$. Note that $SYT(\lambda) = CST(\lambda, n, (1^n))$. We now describe jeu-de-taquin promotion, which is a combinatorial algorithm that gives a function $\partial : CST(\lambda, k) \to CST(\lambda, k)$.

If $T \in CST(\lambda, k)$ does not contain entries equal to k, increase each entry in T by 1 to obtain $\partial(T)$. Otherwise, replace each k in T with a dot. If there is a dot in the figure that is not contained in a continuous strip of dots in the northwest corner, choose the westernmost dot and slide it north or west according to the following diagrams:

$$\begin{array}{|c|c|}\hline a & b \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline \bullet & b \\ \hline a & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ ; \qquad \begin{array}{|c|c|}\hline a & \bullet \\ \hline b & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline \bullet & a \\ \hline b & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ ; \qquad \begin{array}{|c|c|}\hline a & b \\ \hline c & \bullet \\ \hline\end{array} \to \begin{cases} \begin{array}{|c|c|}\hline a & \bullet \\ \hline c & b \\ \hline\end{array} & \text{if } c \le b \\[2ex] \begin{array}{|c|c|}\hline a & b \\ \hline \bullet & c \\ \hline\end{array} & \text{if } b < c. \end{cases}$$

Continue sliding this dot north or west through the figure until it rests in a connected component of dots in the northwest corner. Repeat the procedure for the remaining dots in the figure until all dots belong to a connected component of dots in the northwest corner. Finally, replace all dots with 1s and increase all other entries in the figure by 1 to obtain $\partial(T)$. Jeu-de-taquin promotion preserves the semistandard property (see [13]), and the map $\partial : CST(\lambda, n) \to CST(\lambda, n)$ restricts to a function on $SYT(\lambda)$ which we will denote by $j : SYT(\lambda) \to SYT(\lambda)$.

The group generated by the long cycle $\sigma_k = (1, 2, \ldots, k) \in S_k$ acts on the set of compositions of n of length k by

$$\sigma_k(\alpha_1, \alpha_2, \ldots, \alpha_k) = (\alpha_{\sigma_k^{-1}(1)}, \alpha_{\sigma_k^{-1}(2)}, \ldots, \alpha_{\sigma_k^{-1}(k)}) = (\alpha_k, \alpha_1, \ldots, \alpha_{k-1}).$$

If T has content α, then $\partial(T)$ has content $\sigma_k \alpha$.

Example 2.1. Below is an illustration of jeu-de-taquin promotion on a tableau T in $CST((4, 4, 2), 7)$:

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 5 & 7 \\ \hline 7 & 7 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 5 & \bullet \\ \hline \bullet & \bullet & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline \bullet & 1 & 2 & 3 \\ \hline 1 & 3 & 5 & \bullet \\ \hline 2 & \bullet & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & 2 & 3 \\ \hline 1 & 1 & 5 & \bullet \\ \hline 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$$

$$\to \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & \bullet & 3 \\ \hline 1 & 1 & 2 & 5 \\ \hline 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 4 \\ \hline 2 & 2 & 3 & 6 \\ \hline 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} = \partial(T).$$

Given $T \in CST(\lambda, k)$, the *order of promotion* on T is the least positive integer r with $\partial^r(T) = T$. Given a subset X of $CST(\lambda, k)$ that is invariant under ∂, the *order of promotion* on X is the least positive integer r such that $\partial^r(T) = T$ for all T in X.

Example 2.2. Let $\lambda = (5, 2)$. The following tableaux in $SYT(\lambda)$ have orders 5, 6, and 3, respectively:

$$\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\ \hline 6&7&\multicolumn{3}{c}{}\\ \cline{1-2}\end{array}, \qquad \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&6\\ \hline 5&7&\multicolumn{3}{c}{}\\ \cline{1-2}\end{array}, \qquad \begin{array}{|c|c|c|c|c|}\hline 1&2&3&5&6\\ \hline 4&7&\multicolumn{3}{c}{}\\ \cline{1-2}\end{array}.$$

In fact, $SYT(\lambda)$ contains 14 elements; five have order 5, six have order 6, and three have order 3. The order on $SYT(\lambda)$ is 30.

3. Order of promotion for semistandard hook tableaux

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition of n, and let $n(\alpha)$ denote the number of nonzero entries in α; $n(\alpha) = |\{\alpha_i \neq 0 \mid 1 \leq i \leq k\}|$. Note that, if $\lambda = (n-m, 1^m)$ and $T \in CST(\lambda, k, \alpha)$, then $n(\alpha) > m$, since T has distinct column entries.

Let $\lambda = (n-m, 1^m)$ be a hook partition, and let $\lambda' = (n(\alpha) - m, 1^m)$, which is a partition of $n(\alpha)$. We define a map $\psi_\alpha : CST(\lambda, k, \alpha) \to SYT(\lambda')$ by first deleting repeated entries from the row of $T \in CST(\lambda, k, \alpha)$ and then standardizing the tableau to obtain $\psi_\alpha(T)$. The map ψ_α is defined more formally below.

Define a composition $\beta(\alpha) = (\beta_1, \ldots, \beta_k)$ of $n(\alpha)$ by defining $\beta_i = 1$ if $\alpha_i \neq 0$ and $\beta_i = 0$ if $\alpha_i = 0$. Define

$$\phi_\alpha : CST(\lambda, k, \alpha) \to CST(\lambda', k, \beta(\alpha)), \tag{1}$$

where $\phi_\alpha(T)$ is obtained from T by deleting all entries in the row that are repeated in the row or appear in the column. See Example 3.1 for an illustration.

For $\beta = (\beta_1, \ldots, \beta_k)$ a composition of n with $\beta_i = 0$ or 1, $1 \leq i \leq k$, let

$$\rho : CST(\lambda, k, \beta) \to SYT(\lambda) \tag{2}$$

be the standardization operator. In other words, given a semistandard tableau T with distinct entries, $\rho(T) = S$, where, if $B = (b_1, b_2, \ldots, b_r)$ denotes the sequence of entries in T, written in increasing order, then each b_i is replaced by i, for $1 \leq i \leq r$, in T to form S. Note that ρ is a special case of $std : CST(\lambda, k, \alpha) \to SYT(\lambda)$ given in Rhoades [9].

Define $\psi_\alpha : CST(\lambda, k, \alpha) \to SYT(\lambda')$ as a composition; $\psi_\alpha = \rho \circ \phi_\alpha$. The map ψ_α will allow us to use standard hook tableaux to uncover properties of semistandard hook tableaux.

Example 3.1. Below, we give the image of a tableau $T \in CST((7, 1^3), 9, \alpha)$ under ψ_α, where $\alpha = (0, 2, 2, 1, 1, 1, 1, 0, 2)$:

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline 2&2&3&4&5&6&9\\ \hline 3&\multicolumn{6}{c}{}\\ \cline{1-1} 7&\multicolumn{6}{c}{}\\ \cline{1-1} 9&\multicolumn{6}{c}{}\\ \cline{1-1}\end{array} \xmapsto{\phi_\alpha} \begin{array}{|c|c|c|c|}\hline 2&4&5&6\\ \hline 3&\multicolumn{3}{c}{}\\ \cline{1-1} 7&\multicolumn{3}{c}{}\\ \cline{1-1} 9&\multicolumn{3}{c}{}\\ \cline{1-1}\end{array} \xmapsto{\rho} \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\ \hline 2&\multicolumn{3}{c}{}\\ \cline{1-1} 6&\multicolumn{3}{c}{}\\ \cline{1-1} 7&\multicolumn{3}{c}{}\\ \cline{1-1}\end{array} = \psi_\alpha(T).$$

Since ρ and ϕ_α are both bijections, we have the following theorem.

Theorem 3.2. *Let $\lambda = (n-m, 1^m)$ be a hook partition, and let α be a composition of n. Then $|CST(\lambda, k, \alpha)| = \binom{n(\alpha)-1}{m}$.*

Let $\sigma_k = (1, 2, \ldots, k)$ denote the long cycle in S_k, and let α be a composition of n. We say that α has *cyclic symmetry* $p(\alpha)$ if $p(\alpha)$ is the smallest positive integer such that $\sigma_k^{p(\alpha)}\alpha = \alpha$. Then $1 \leq p(\alpha) \leq k$ and $\alpha_i = \alpha_j$ whenever $i \equiv j \mod p(\alpha)$. Define $f(\alpha)$ to be the number of nonzero entries in the first $p(\alpha)$ entries of α:

$$f(\alpha) = |\{\alpha_i \neq 0 \mid 1 \leq i \leq p(\alpha)\}|.$$

Note that $f(\alpha)$ is the number of nonzero entries in any sequence of $p(\alpha)$ consecutive entries of α. Since $p(\alpha)$ divides k and $\frac{k \cdot f(\alpha)}{p(\alpha)} = n(\alpha)$, $f(\alpha)$ divides $n(\alpha)$.

If $T \in CST(\lambda, k, \alpha)$, then $\partial(T)$ has content $\sigma_k\alpha$ and $\partial^r(T)$ has content $\sigma_k^r\alpha$ for any positive integer r. Thus, if $T \in CST(\lambda, k, \alpha)$, where α has cyclic symmetry $p(\alpha)$, then $\partial^{p(\alpha)}(T)$ has content α, so we have a map $\partial^{p(\alpha)} : CST(\lambda, k, \alpha) \to CST(\lambda, k, \alpha)$. The following example motivates our first theorem.

Example 3.3. Consider $T = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&4&5&7&8\\ \hline 3&\multicolumn{5}{c}{}\\ \cline{1-1} 5&\multicolumn{5}{c}{}\\ \cline{1-1}\end{array} \in CST((6, 1^2), 8, \alpha)$, where $\alpha = (2, 0, 1, 1, 2, 0, 1, 1)$. Then $p(\alpha) = 4$ and $f(\alpha) = 3$. We have

$$\psi_\alpha \circ \partial^4(T) = \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\ \hline 2&\multicolumn{3}{c}{}\\ \cline{1-1} 4&\multicolumn{3}{c}{}\\ \cline{1-1}\end{array} = j^3 \circ \psi_\alpha(T).$$

Theorem 3.4. *Suppose that $\lambda = (n-m, 1^m)$, and let T belong to $CST(\lambda, k, \alpha)$, where α has cyclic symmetry $p(\alpha)$. Then*

$$\psi_\alpha \circ \partial^{p(\alpha)}(T) = j^{f(\alpha)} \circ \psi_\alpha(T),$$

where ∂ denotes the jeu-de-taquin promotion operator on $CST(\lambda, k, \alpha)$ and j the jeu-de-taquin promotion operator on $SYT(\lambda')$, where $\lambda' = (n(\alpha) - m, 1^m)$.

Proof. Denote the long cycle σ_k in S_k by σ, the cyclic symmetry of α by p, and let

$$\psi_0 = \psi_\alpha, \psi_1 = \psi_{\sigma\alpha}, \ldots, \psi_i = \psi_{\sigma^i\alpha}, \ldots, \psi_p = \psi_{\sigma^p\alpha} = \psi_0.$$

We have $\psi_i = \rho \circ \phi_{\sigma^i\alpha}$, where ρ and $\phi_{\sigma^i\alpha}$ are as defined in (1) and (2). We will write $\phi_{\sigma^i\alpha} = \phi_i$.

Let f_i denote the number of nonzero entries in the last i entries of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ for $0 \le i \le p$; note that $f_p = f(\alpha)$. We will prove by induction that $\psi_i \circ \partial^i(T) = j^{f_i} \circ \psi_0(T)$ for $1 \le i \le p$.

We first claim that

$$\psi_1 \circ \partial(T) = \begin{cases} \psi_0(T) & \text{if } \alpha_k = 0 \\ j \circ \psi_0(T) & \text{if } \alpha_k \neq 0. \end{cases}$$

If $\alpha_k = 0$, then $\partial(T) = S$, where the entries of S are obtained by increasing each entry of T by 1. Since this does not affect the ordering of the entries of S (in other words, if $T(i_1, j_1) \le T(i_2, j_2)$ in T then $S(i_1, j_1) \le S(i_2, j_2)$ in S), we have $\psi_1(S) = \psi_0(T)$.

Suppose that $\alpha_k \neq 0$. We have $\psi_1 \circ \partial(T) = \rho \circ \phi_1 \circ \partial(T)$ and $j \circ \psi_0(T) = j \circ \rho \circ \phi_0(T) = \rho \circ \partial \circ \phi_0(T)$, so we will prove that $\rho \circ \phi_1 \circ \partial(T) = \rho \circ \partial \circ \phi_0(T)$. Both $\phi_1 \circ \partial(T)$ and $\partial \circ \phi_0(T)$ have content $\sigma\beta(\alpha)$, and each is determined by its column. But ϕ_1 does not alter the column of $\partial(T)$, so the column of $\phi_1 \circ \partial(T)$ is the same as the column of $\partial(T)$. Similarly, the column of $\phi_0(T)$ is equal to that of T, so the column of $\partial \circ \phi_0(T)$ is equal to that of $\partial(T)$.

Suppose that $\psi_i \circ \partial^i(T) = j^{f_i} \circ \psi_0(T)$. The content of $\partial^i(T)$ is equal to $\sigma^i\alpha$. If the kth entry in $\sigma^i\alpha$ is equal to zero, then $\alpha_{k-i} = 0$ and $f_i = f_{i+1}$. In this case, ∂ applied to $\partial^i(T)$ simply increases each entry by 1. As above,

$$\psi_{i+1} \circ \partial^{i+1}(T) = \psi_i \circ \partial^i(T) = j^{f_i} \circ \psi_0(T) = j^{f_{i+1}} \circ \psi_0(T).$$

If the kth entry of $\sigma^i\alpha$ is not equal to zero, then $f_{i+1} = f_i + 1$. Then

$$j^{f_{i+1}} \circ \psi_0(T) = j^{f_i+1} \circ \psi_0(T) = j \circ \psi_i \circ \partial^i(T) = j \circ \rho \circ \phi_i \circ \partial^i(T).$$

As above, $j \circ \rho \circ \phi_i \circ \partial^i(T) = \rho \circ \partial \circ \phi_i \circ \partial^i(T)$, and

$$\psi_{i+1} \circ \partial^{i+1}(T) = \rho \circ \phi_{i+1} \circ \partial^{i+1}(T).$$

Since $\phi_{i+1} \circ \partial^{i+1}(T) = \partial \circ \phi_i \circ \partial^i(T)$, we have

$$\psi_{i+1} \circ \partial^{i+1}(T) = \rho \circ \partial \circ \phi_i \circ \partial^i(T) = j^{f_{i+1}} \circ \psi_0(T). \quad \square$$

Remark 3.5. The set of standard hook tableaux $SYT(n-m, 1^m)$ is in one-to-one correspondence with the set B of m-element subsets of $\{2, \ldots, n\}$ via the bijection $\theta : SYT(\lambda) \to B$ that sends a tableau T to the set corresponding to the entries in the column of T below the entry in the $(1, 1)$-box. Furthermore, jeu-de-taquin promotion on $SYT(\lambda)$ corresponds to the action of $\sigma = (2, 3, \ldots, n)$ on B, so $\theta(j(T)) = \sigma(\theta(T))$. It follows that, for $m \ge 1$ and $n - m \ge 2$, the order of promotion on $SYT(n-m, 1^m)$ is equal to $n - 1$.

Theorem 3.6. *Let $\lambda = (n-m, 1^m)$, and let T in $CST(\lambda, k, \alpha)$, where α has cyclic symmetry $p(\alpha)$. The order of promotion of T is equal to $p(\alpha) \cdot t$, where t is the order of $\psi_\alpha(T)$.*

Proof. By Theorem 3.4, $\psi_\alpha \circ \partial^{p(\alpha)t}(T) = j^{f(\alpha)t} \circ \psi_\alpha(T) = \psi_\alpha(T)$. Since ψ_α is a bijection, $\partial^{p(\alpha)t}(T) = T$.

Suppose that $\partial^r(T) = T$ for some positive integer r. Then $r = p(\alpha)y$ for some positive integer y. Since $\psi_\alpha(T) = \psi_\alpha \circ \partial^{p(\alpha)y}(T) = j^{f(\alpha)y} \circ \psi_\alpha(T)$ and the order of $\psi_\alpha(T)$ is equal to t, it must be that t divides $f(\alpha) \cdot y$.

But $\psi_\alpha(T) \in SYT(\lambda')$, where $\lambda' = (n(\alpha) - m, 1^m)$ is a hook partition of $n(\alpha)$, so t also divides $n(\alpha) - 1$. Since $f(\alpha)$ divides $n(\alpha)$, $gcd(t, f(\alpha)) = 1$. It follows that t divides y, so $p(\alpha) \cdot t$ divides r, and the order of promotion of T is indeed equal to $p(\alpha) \cdot t$. $\square$

Note that, if $\lambda = (n-m, 1^m)$ and T is a λ-tableau with content equal to α, then $n(\alpha) \ge m + 1$. If λ is a one-column partition, then $SYT(\lambda)$ contains only one tableau, so the order of promotion on $SYT(\lambda)$ is 1. The following corollary follows from Theorem 3.6.

Corollary 3.7. *Let $m \ge 1$ and $n - m \ge 2$, and let $\lambda = (n-m, 1^m)$ be a hook partition of n. The order of promotion on $CST(\lambda, k, \alpha)$ is equal to $p(\alpha)$ if $n(\alpha) = m + 1$, and $p(\alpha) \cdot (n(\alpha) - 1)$ otherwise.*

Example 3.8. Consider $CST((6, 1^4), 9, \alpha)$, where $\alpha = (2, 0, 1, 1, 2, 0, 1, 1, 2, 0, 1, 1)$. We have

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 & 6 & 8 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1} 7 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array} \quad \xmapsto{\psi_\alpha} \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 6 & 8 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1} 7 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array} = S.$$

Since S has order 2, and $p(\alpha) = 3$, T has order 6, by Theorem 3.6. The order of promotion on $CST((6, 1^4), 9, \alpha)$ is equal to 18.

By considering the various contents α of tableaux T that belong to $CST(\lambda, k)$, we obtain the order of promotion on $CST(\lambda, k)$.

Theorem 3.9. *Let $\lambda = (n - m, 1^m)$ be a hook partition of n with $m \geq 1$, $n - m \geq 2$. The order of promotion on $CST(\lambda, k)$ is equal to the following:*

$$\begin{cases} k & \text{if } k = m + 1, \\ k \cdot \mathrm{lcm}(m + 1, m + 2, \ldots, k - 2, k - 1) & \text{if } m + 1 < k < n, \\ k \cdot \mathrm{lcm}(m + 1, m + 2, \ldots, n - 2, n - 1) & \text{if } k \geq n. \end{cases}$$

Proof. Suppose that $k = m + 1$, and let $T \in CST(\lambda, k)$ have content α. Then $n(\alpha) = m + 1$, since T would have at least one repeat in its column otherwise. By Corollary 3.7, the order of T divides $p(\alpha)$, which divides k. If T has content $\alpha = (n-m, 1^m)$ then $p(\alpha) = k$ so T has order k. Consequently, $CST(\lambda, k)$ has order k when $k = m + 1$.

If $m+1 < k < n$ and $T \in CST(\lambda, k)$ has content α, then $m+1 \leq n(\alpha) \leq k$. If $n(\alpha) = m+1$, then, as above, the order of T divides k, and there is at least one element in $CST(\lambda, k)$ with $n(\alpha) = m+1$ and order equal to k. Suppose that $n(\alpha) = m+\ell$, where $2 \leq \ell \leq k-m$. By Corollary 3.7, the order of T then divides $p(\alpha)\cdot(m+\ell-1)$, which divides $k\cdot(m+\ell-1)$. Moreover, any tableau with content $\alpha = (n-(m+\ell-1), 1^{m+\ell-1}, 0^{k-(m+\ell)})$ has order $k\cdot(m+\ell-1)$, since $p(\alpha) = k$ in this instance. It follows that the order of promotion on $CST(\lambda, k)$ is equal to $\mathrm{lcm}(k, k(m+1), k(m+2), \ldots, k(k-1)) = k\cdot\mathrm{lcm}(m+1, m+2, \ldots, k-1)$.

Suppose that $k > n$, so that $m + 1 \leq n(\alpha) \leq n$. The case where $n(\alpha) = m + 1$ is covered above. Let $n(\alpha) = m + \ell$, where $2 \leq \ell \leq n - m$. Any tableau in $CST(\lambda, k, \alpha)$ has order that divides $k \cdot (m + \ell - 1)$, and a tableau with content $\alpha = (n - (m + \ell - 1), 1^{m+\ell-1}, 0^{k-(m+\ell)})$ has order $k \cdot (m + \ell - 1)$. The order of promotion on $CST(\lambda, k)$ is thus equal to $\mathrm{lcm}(k, k(m+1), \ldots, k(n-2), k(n-1)) = k \cdot \mathrm{lcm}(m+1, m+2, \ldots, n-2, n-1)$. In the case where $k = n$, the argument is the same except when $n(\alpha) = n$, in which case $\alpha = (1^n)$, and the order of promotion on $CST(\lambda, k, \alpha)$ is $n - 1$. The order of promotion on $CST(\lambda, k)$ in this case is $\mathrm{lcm}(k, k(m+1), \ldots, k(n-2), n-1) = k \cdot \mathrm{lcm}(m+1, m+2, \ldots, n-2, n-1)$. $\square$

Example 3.10. Let $\lambda = (5, 1^2)$. If $k = 5$, the order of promotion on $CST(\lambda, k)$ is equal to 60. When $k = 7$, the order of promotion on $CST(\lambda, k)$ increases to $420 = 7 \cdot \mathrm{lcm}(6, 5, 4, 3)$.

Remark 3.11. The least common multiple of a finite number of consecutive integers was studied by Farhi [2]; he showed that $\mathrm{lcm}(x, x + 1, \ldots, x + \ell)$ is a multiple of $x\binom{x+\ell}{\ell}$. If $x(x + \ell) \equiv 0 \mod (\ell!)$, then $\mathrm{lcm}(x, x + 1, \ldots, x + \ell)$ is equal to $x\binom{x+\ell}{\ell}$, as in the above example.

4. A cyclic sieving phenomenon for semistandard hook tableaux

Let X be a finite set, and let $C = \langle g \rangle$ be a cyclic group of order N that acts on X. Given any element $g^d \in C$, denote its fixed point set by X^{g^d}. Let ω be a primitive Nth root of unity, and let $X(q) \in \mathbb{Z}[q]$. The triple $(X, C, X(q))$ exhibits the *cyclic sieving phenomenon* (CSP) if, for any integer $d \geq 0$, $X(\omega^d) = |X^{g^d}|$.

Since $X(1) = |X|$, $X(q)$ is generally found by taking a q-deformation of a formula that gives $|X|$. Define the *q-analogue of n* by

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Define $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ and the *q-binomial coefficients*, which are polynomials with integer coefficients [13], by $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$.

Theorem 4.1 (*Reiner, Stanton and White [8]*). *Let X denote the set of k-element subsets of $\{1, 2, \ldots, n\}$, let the cyclic group $C = \mathbb{Z}/n\mathbb{Z}$ act on X via the long cycle $(1, 2, \ldots, n) \in S_n$, and let $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$. Then $(X, C, X(q))$ exhibits the CSP.*

Jeu-de-taquin promotion $\partial : CST(\lambda, k) \to CST(\lambda, k)$ generates an action of $\mathbb{Z}/t\mathbb{Z}$ on $CST(\lambda, k)$, where t is the order of jeu-de-taquin promotion on $CST(\lambda, k)$. We will use the following corollary to Theorem 4.1, which follows from Remark 3.5, to give a CSP for $CST(\lambda, k, \alpha)$, where $\lambda = (n - m, 1^m)$ is a hook partition.

Corollary 4.2. *Let $\lambda = (n - m, 1^m)$ be a hook partition of n, and let $C = \mathbb{Z}/(n-1)\mathbb{Z}$ act on $X = SYT(\lambda)$ by jeu-de-taquin promotion. Then the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where $X(q) = \begin{bmatrix} n-1 \\ m \end{bmatrix}_q$.*

Let α be a composition of k, suppose that α has cyclic symmetry $p = p(\alpha)$, and consider $\partial^p : CST(\lambda, k, \alpha) \to CST(\lambda, k, \alpha)$. Since the order of promotion on $CST(\lambda, k, \alpha)$ is equal to $p \cdot (n(\alpha) - 1)$ when $\lambda = (n - m, 1^m)$, ∂^p gives an action of the cyclic group $\mathbb{Z}/(n(\alpha) - 1)\mathbb{Z}$ on $CST(\lambda, k, \alpha)$.

Theorem 4.3. *Let $X = CST(\lambda, k, \alpha)$, where $\lambda = (n - m, 1^m)$, and suppose that α has cyclic symmetry $p(\alpha) = p$. Let $C = \mathbb{Z}/(n(\alpha) - 1)\mathbb{Z}$ act on X by the pth power of promotion. Then the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where $X(q) = \begin{bmatrix} n(\alpha)-1 \\ m \end{bmatrix}_q$.*

Proof. Let $\lambda' = (n(\alpha) - m, 1^m)$. We first show that, for any positive integer r, the number of tableaux in $CST(\lambda, k, \alpha)$ that are fixed by ∂^{pr} is equal to the number of tableaux in $SYT(\lambda')$ that are fixed by j^r.

Suppose that $\partial^{pr}(T) = T$ for $T \in CST(\lambda, k, \alpha)$. Then $\psi_\alpha \circ \partial^{pr}(T) = \psi_\alpha(T)$, so, by Theorem 3.4, $j^{f(\alpha)r} \circ \psi_\alpha(T) = \psi_\alpha(T)$. Let s denote the order of $\psi_\alpha(T)$. Then s divides both $f(\alpha) \cdot r$ and $n(\alpha) - 1$. Since $f(\alpha)$ divides $n(\alpha)$, $\gcd(s, f(\alpha)) = 1$. Thus s divides r, so $j^r \circ \psi_\alpha(T) = \psi_\alpha(T)$, and there are at least as many tableaux in $CST(\lambda, k, \alpha)$ that are fixed by ∂^{pr} as there are tableaux in $SYT(\lambda')$ that are fixed by j^r. If $j^r(S) = S$ for $S \in SYT(\lambda')$ and $\psi_\alpha(T) = S$, then $j^{rf(\alpha)} \circ \psi_\alpha(T) = \psi_\alpha(T)$, and, by Theorem 3.4, $\psi_\alpha \circ \partial^{pr}(T) = \psi_\alpha(T)$. Thus $\partial^{pr}(T) = T$.

By Corollary 4.2, there are $X(\omega^r)$ tableaux in $SYT(\lambda')$ that are fixed by j^r, where ω is a primitive tth root of unity for $t = n(\alpha) - 1$. The result now follows. $\square$

Example 4.4. Let $\lambda = (6, 1^2)$, $k = 5$, and $\alpha = (2, 1, 2, 1, 2)$. Then $|CST(\lambda, k, \alpha)| = 6$, and, since $p(\alpha) = 5$, we let $C = \langle \partial^5 \rangle$. We have $n(\alpha) - 1 = 4$ and

$$X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4.$$

If ω is a primitive fourth root of unity,

$$|X^{\partial^5}| = X(\omega) = 0, \qquad |X^{\partial^{10}}| = X(\omega^2) = 2, \qquad |X^{\partial^{15}}| = X(\omega^3) = 0, \qquad |X^{\partial^{20}}| = X(1) = 6.$$

The following two tableaux in $CST(\lambda, k, \alpha)$ have order 10:

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 4 & 5 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}\;, \qquad \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 5 & 5 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}\;.$$

All other tableaux in $CST(\lambda, k, \alpha)$ have order 20.

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